

Lecture notes on Fermi Liquids

Contents

1	Introduction and experimental observations	2
1.1	^3He	2
1.2	Heavy fermions in CeAl_3	3
2	Landau quasiparticles	5
3	Signatures of a Fermi liquid and Landau parameters	8
3.1	Specific heat	9
3.2	Spin susceptibility	10
4	Microscopic basis for the Fermi Liquid theory	10
4.1	Non-interacting Green's function and spectral function	11
4.2	Interacting Green's function and spectral function	12
4.3	Imaginary part of the self-energy: lifetime	13
4.4	Real part of the self-energy: effective mass and quasiparticle residue	13

1 Introduction and experimental observations

The specific heat is a measurable quantity that relates the heat (energy) added to a physical system to a change in its temperature. For a Fermi gas this is entirely determined by the thermal excitation of free electrons above the Fermi energy in an energy window of the order $k_B T$

$$C = \frac{1}{V} \left(\frac{\partial E}{\partial T} \right)_V = \frac{\pi^2}{3} k_B T g(\epsilon_F) \quad (1)$$

This fact is a consequence of the Fermi statistics and of the Pauli principle for fermions. It is important to note that the specific heat of a non-interacting Fermi gas has a linear temperature dependence

$$C = \gamma T \quad (2)$$

The linear slope is a measure of the density of states at the Fermi energy $g(\epsilon_F)$, that is related to the effective mass of the electrons.

1.1 ^3He

^3He is characterized by a spin quantum number $1/2$, unlike the more common isotope ^4He ($s = 0$). Therefore, by virtue of the spin-statistics theorem, it obeys the Fermi statistics. The low-temperature liquid state of ^3He has been studied for many years as a prototype of a fluid of interacting fermions in the absence of complications introduced by the crystalline environment of solids. In particular, in ^3He we find full translational and rotational symmetries which greatly simplify its analysis. Moreover, since ^3He is charge neutral, the interaction between atoms can be described as short range, avoiding all the complications introduced by the screening of the long-range Coulomb interaction in metals.

The temperature-pressure phase diagram of ^3He is shown in Figure 1. From this diagram we take home that 1) ^3He remains a normal liquid (non superfluid) down to the mK range and 2) ^3He can be turned into a solid applying a pressure of about 30 bar. This transition towards a solid state offers the opportunity to enhance the strength of the interaction between fermions by applying pressure.

Figure 2 shows the specific heat of ^3He as a function of temperature measured at two different pressures (Greywall et al. Phys. Rev. B 27, 2747 (1983)). The low-temperature linear dependence of this quantity (highlighted by the dashed lines) indicates something very important: in this regime ^3He exhibits the same qualitative behaviour of a free electron gas (see equation 1). On the other hand, the slope of the linear approximation reveal a higher effective mass than the bare mass of ^3He , being on the order of $5m_{He3}$. This fact does not find an explanation within a free Fermi gas approximation. Please note that we have realised that interactions with the crystalline lattice in a quasi-realistic band picture (for example our tight-binding toy model of lecture 1) lead to variations of the group velocity of electronic waves and therefore a renormalisation of density of states and effective masses. Here we are experimentally confronted with something different, a much stranger

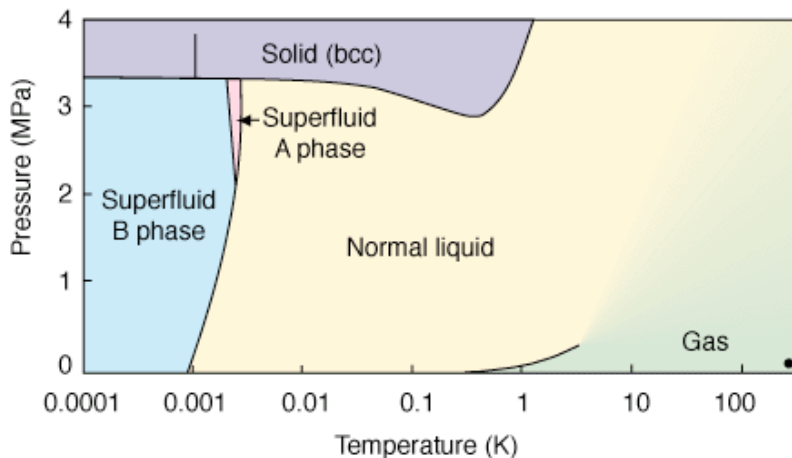


Figure 1: Temperature-pressure phase diagram of ^3He . From l1.tkk.fi

beast. In a translationally invariant environment with tunable interactions we observe the persistence of a single-particle-like thermal response with a renormalised density of states (effective mass). Understanding the qualitative similarities but quantitative differences with a non-interacting picture is one of the main goals of our discussion.

1.2 Heavy fermions in CeAl_3

Figure 3 shows the specific heat of an intermetallic compound, CeAl_3 (Phys. Rev. Lett. 35, 1779 (1975)). As discussed previously in the case of ^3He , this material exhibits also a linear temperature dependence of the specific heat. The slope of the linear regression though, points to an effective mass that is about 1000 times larger than the mass of free electrons! For this reason, this material and related compounds are known as heavy fermions. Although the electron mass undergoes such a large renormalization, the material still behaves qualitatively as a free electron gas. By comparing the previous case of ^3He with CeAl_3 we are persuaded that the same phenomenology holds in very different physical realizations of interacting systems of fermions with a large tolerance to the renormalization of the effective masses. Therefore there must be a very profound and robust mechanism at work. We will see that this is indeed the case and this mechanism is known as the formation of Landau quasiparticles.

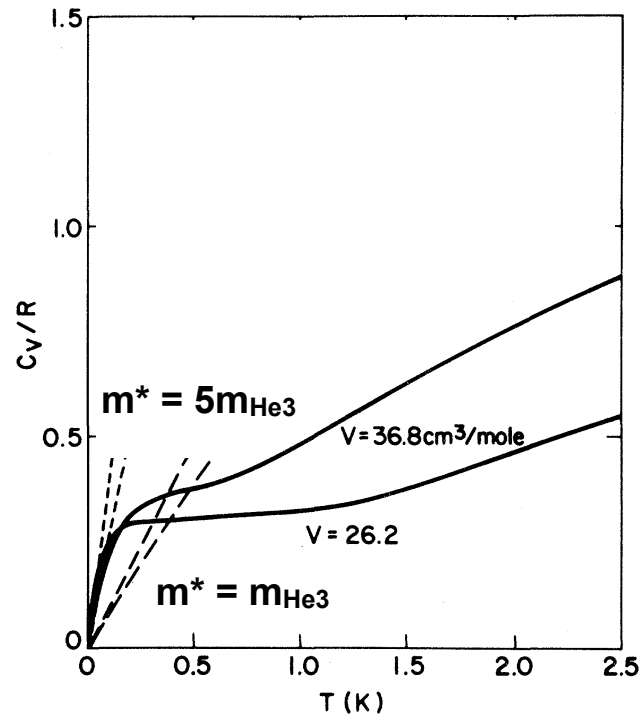


Figure 2: Solid lines: specific heat of ^3He as a function of temperature measured at two different pressures. Dashed lines: linear dependence expected for a given effective mass. From Greywall et al. Phys. Rev. B 27, 2747 (1983)

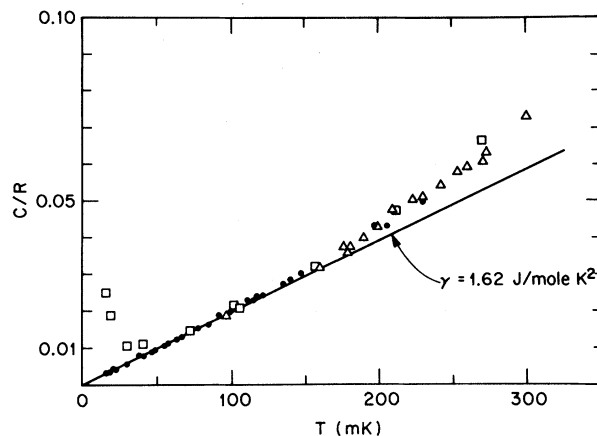


Figure 3: . Specific heat as a function of temperature of CeAl_3 . From Phys. Rev. Lett. 35, 1779 (1975)

2 Landau quasiparticles

We begin by outlining the *basic assumptions* of the Landau theory of Fermi liquids. As we have seen in our first lecture, the elementary excitations of the Fermi gas are: 1) single-electron excitations outside the Fermi surface ($k > k_F$) or single-hole excitations inside the Fermi surface ($k < k_F$). We can describe both these elementary excitations of the Fermi gas as a single, energy-momentum (dispersion) relation:

$$|\epsilon(k)| = \frac{\hbar^2 |k^2 - k_F^2|}{2m} \simeq \hbar v_F |k - k_F| \quad (3)$$

These excitations are probed in a variety of experiments based on the addition or removal of single-particles from many-body systems such as photoemission spectroscopy and scanning tunnelling spectroscopy.

Now we turn on electron-electron interactions and we represent this process as an adiabatic increase of a potential energy λV by means of a slow increase of the parameter $\lambda \in [0, 1]$. We *assume* that the spectrum of eigenstates of the interacting hamiltonian is adiabatically connected to the non-interacting Fermi gas, i.e. to each ground state wavefunction we can assign a corresponding interacting quantum state labeled by the same set of quantum numbers. This condition is not verified whenever we have a transition to a different state of matter driven by interactions (e.g. superconductivity and other Fermi liquid instabilities introduced in the previous lecture, often characterised by an absence of single-particle density of states at the Fermi surface, within an energy range called the energy gap). However, it is a good approximation for a vast number of many-body systems.

Provided that the adiabatic condition holds, we can restate the fundamental relationship between the electron density and the k-space volume occupied by a system of fermions. Following our definition of Green's function, the particle density n is in relation with the volume in k-space inside of which $G(\omega = 0, k) > 0$ (D is the dimensionality of the system)

$$n = 2 \int_{G(\omega=0,k)>0} \frac{d^D k}{(2\pi)^D} \quad (4)$$

In a system with a well-defined Fermi surface, the Green's function diverges at its edge. Within the Fermi surface, we find a k-space volume where the Green's function is positive. After diverging at the edge of this volume, the Green's function becomes negative outside. In this condition we may formulate the statement "the density of particles is proportional to the volume inside the Fermi surface" or

$$n = \frac{k_F^3}{3\pi^2} \quad (5)$$

In the presence of interactions that can be described by perturbation theory (do not lead to a phase transition or the insurgence of a gap at the Fermi surface), the Luttinger's theorem states that *this relationship continues to be valid at any order of the perturbative expansion.*

We are interested in describing finite temperature properties of interacting systems of fermions, therefore we necessarily need to address the possibility of an adiabatic connection between excited states of the Fermi gas and excited states of an interacting electron system.

If the many-body system has a well-defined Fermi surface, then we can make the following considerations on the nature of the excited states in the adiabatic approximation. We will make an intuitive (phase space) argument due to Landau, that allows us to understand the nature of the scattering processes without resorting to complex calculations.

Let's consider a filled Fermi sphere and an extra electron added to the system with momentum k and energy $\epsilon(k) = \epsilon_1 > \epsilon_F$. This electron interacts with the Fermi sea, resulting in the creation of a particle-hole excitation. As the additional electron changes its momentum to $k - q$, a hole appears in k' and an electron with momentum $k' + q$ is promoted outside the Fermi surface. Conservation of energy and momentum apply, resulting in

$$k = k - q - k' + k' + q \quad (6)$$

and

$$\epsilon(k) = \epsilon(k - q) - \epsilon(k') + \epsilon(k' + q) \quad (7)$$

Please note that in principle a large number of such scattering channels are possible and it looks like the excitation should readily decompose into a broad spectrum of excited states, losing its identity as a single-particle excitation. However, the limited phase space available plays an important role that we will now outline.

The scattering rate of these processes is proportional to the matrix elements of the interaction $|V(q)|^2$, the number of available states, bounded by conservation of momentum and energy (Fermi golden rule), summing over all possible configurations for k' (hole states) and q (exchanged momentum):

$$\Gamma(k) = \frac{2\pi}{\hbar} \frac{1}{\gamma/2} \sum_{k'q} |V(q)|^2 f(\epsilon_{k'}) [1 - f(\epsilon_{k-q})] [1 - f(\epsilon_{k'+q})] \delta(\epsilon(k-q) - \epsilon(k') + \epsilon(k'+q) - \epsilon(k)) \quad (8)$$

where we have indicated with the usual notation f the distribution function (see notes to lecture 1). The factor $f(\epsilon_{k'})$ ensures the availability of a hole state with momentum $-k'$, while $[1 - f(\epsilon_{k-q})][1 - f(\epsilon_{k'+q})]$ ensures the availability of an electronic state to scatter into by exchanging momentum q .

We will not perform this integration formally but we can intuitively see that the Pauli principle (no double occupation of states) and the sharp decrease of the Fermi distribution at ϵ_F constrains the available states for scattering in two factors:

1) Phase space integrated by k' : the shell of electrons within the Fermi sea available for interactions within the constraints of energy conservation and Pauli principle are limited to a top layer of the Fermi surface with energy range $(\epsilon_1 - \epsilon_F)$ (k' needs to be close enough to the Fermi surface in order to promote an extra electron outside the Fermi sea).

2) Phase space integrated by q : the shell of final states available for occupation is also limited to an energy range $(\epsilon_1 - \epsilon_F)$ ($k' + q$ needs to be outside of the Fermi surface in order to contribute to scattering).

Therefore the scattering rate is expected to show an energy dependence

$$\Gamma \propto (\epsilon_1 - \epsilon_F)^2 \propto \omega^2 \quad (9)$$

Formal calculations confirm indeed this intuitive result. Please reflect on its meaning: as we consider interacting electrons closer and closer to the Fermi surface ($\omega \rightarrow 0$), we expect to see an increased lifetime in the presence of electron-electron scattering. The farther we move away from the Fermi surface the faster scattering processes will manifest.

We are getting close to the solution of our outstanding problem. We want to establish whether or not it is possible to describe the excitations of an interacting Fermi system as quasi-adiabatically connected with the spectrum of excited states of the non-interacting Fermi gas. In order to identify these as legitimate particles with a quasi-adiabatic connection with the non-interacting Fermi gas, we still need to tackle one pressing issue. Since our excitations are characterized by a finite lifetime, in order to identify them as particles *with a well-defined energy*, we need to make sure that their wavefunction exhibit several oscillations before damping. If the damping produced by the finite lifetime is fast compared to the oscillations period, the excitation does not have a well-defined energy. In formula

$$\epsilon(k)^{-1} \gg \tau \quad (10)$$

At first sight this requirement seems very hard to meet. Using the Landau phase space argument outlined in the previous section, we realize that *the scattering rate grows quadratically in energy as a consequence of the discontinuity of the Fermi statistics and the Pauli principle.*

$$\tau^{-1}(\omega) \propto \omega^2 \quad (11)$$

Since the period T of the oscillation of the wavefunction grows as

$$T^{-1} \propto \omega \quad (12)$$

we can observe that

$$\frac{\tau}{T} \propto \omega^{-1} \rightarrow \infty, \omega \rightarrow 0 \quad (13)$$

As we approach the Fermi energy, the single particle excitations acquire a well-defined energy as the increase of their lifetime is more rapid than the decrease in oscillation frequency. The phase space available for scattering processes is reduced very rapidly as we approach the Fermi energy, making these excitations long-lived particles.

This important observation consolidates our interpretation of the single-particle excitations of quantum liquids as legitimate particles. These particle-like excitations are known as *quasiparticles*. Systems of interacting fermions that display such properties are known as *Fermi Liquids*.

3 Signatures of a Fermi liquid and Landau parameters

A hallmark of electron-electron interactions described by a Fermi liquid state is a depletion of the states below the Fermi energy (through electron-hole excitations) and a population of states above the Fermi energy. The electron distribution is therefore modified from a step function to a function that is still discontinuous at the Fermi energy but decreases smoothly on both sides of the discontinuity. The size of the discontinuity is not 1, as in the case of the free Fermi gas but it is identified as a quantity, known as the quasiparticle residue $Z_k < 1$, defined below. Another important signature of a Fermi liquid state is the nature of the temperature dependence of the resistivity, a T^2 law, discussed in the following lecture.

The Landau theory goes on to develop the specific heat, effective mass, compressibility, spin susceptibility (Landau parameters) of interacting electrons. Collective phenomena through quasiparticles interactions can even be described and these will be addressed in lecture 9.

Based on our definition of quasiparticles outlined above, we are in a position to write an expression for the free energy F , as a functional of the quasiparticle distribution $\delta n_{\mathbf{k},\sigma}$. From the free energy we will be able to compute the thermodynamics responses of interest for a Fermi liquid, such as specific heat and spin susceptibility.

$$F[\delta n_{\mathbf{k},\sigma}] = \underbrace{E_{\text{GS}}}_{\text{ground state energy}} + \underbrace{\sum_{\mathbf{k},\sigma} \epsilon(\mathbf{k}) \delta n_{\mathbf{k},\sigma}}_{\text{renormalized kinetic energy}} + \underbrace{\frac{1}{2} \sum_{\mathbf{k},\sigma,\mathbf{k}',\sigma'} f_{\mathbf{k},\sigma,\mathbf{k}',\sigma'} \delta n_{\mathbf{k},\sigma} \delta n_{\mathbf{k}',\sigma'}}_{\text{quasiparticle-quasiparticle interactions}} - TS \quad (14)$$

In the kinetic energy term we find a renormalised dispersion

$$\epsilon(\mathbf{k}) \simeq \frac{k_F}{m^*} (\mathbf{k} - \mathbf{k}_F) - \mu \quad (15)$$

parameterised with an effective mass m^* . This reflects the dressing of the quasiparticles by interactions. The quasiparticle-quasiparticle interactions are described by the interaction term $f_{\mathbf{k},\sigma,\mathbf{k}',\sigma'}$ that can be decomposed in a spin symmetric and a spin antisymmetric component $f^{s,a}$

$$f^{s,a} = (f_{\uparrow\uparrow} \pm f_{\uparrow\downarrow}) / 2 \quad (16)$$

Because of rotational invariance, $f^{s,a}$ only depends on the angle θ between \mathbf{k} and \mathbf{k}' , i.e. $f^{s,a}(\cos \theta)$. These functions can be expressed as an expansion in Legendre polynomials $P_l(\cos \theta)$.

$$f^{s,a} = \frac{1}{2g^*(\epsilon_F)} \sum_{l=0}^{\infty} (2l+1) F_l^{s,a} P_l(\cos \theta) \quad (17)$$

The coefficients of this expansion $F_l^{s,a}$ are called the Landau parameters and they are defined below:

$$F_l^{s,a} = 2g^*(\epsilon_F) \int \frac{2\pi \sin \theta d\theta}{4\pi} f^{s,a} P_l(\cos \theta) \quad (18)$$

Using the definitions proposed here, the Landau coefficients are adimensional, providing a useful characterisation of the strength of the renormalisation of the thermodynamic properties engendered by interactions. In practice the expansion is limited to the first few terms. Below we summarise the results of the effects on the renormalisation of the specific heat and spin susceptibility.

3.1 Specific heat

The Landau coefficient F_1^s defines the renormalisation of the quasiparticle mass, influencing various thermodynamic probes sensitive to the density of states, such as the specific heat.

$$m^* = m(1 + F_1^s) \quad (19)$$

The Free energy of a Fermi liquid outlined above leads to a specific heat of the form:

$$C = \frac{\pi^2}{3} k_B T g^*(\epsilon_F) = \frac{m^* k_F}{3\hbar^2} k_B T = \frac{m(1 + F_1^s) k_F}{3\hbar^2} k_B T \quad (20)$$

3.2 Spin susceptibility

An external magnetic field will split the Fermi surface of a Fermi gas in spin up and spin down electrons, by means of the Zeeman interaction. Similarly, in a Fermi liquid the total volume of the Fermi surface is conserved and the densities of quasiparticles are split in a up and down spin component according to the susceptibility as defined by the Landau parameter F_0^a :

$$\chi_S = \chi_S^0 \frac{1 + F_1^s}{1 + F_0^a} \quad (21)$$

In this expression χ_S^0 is the Pauli paramagnetic susceptibility of the Fermi gas.

4 Microscopic basis for the Fermi Liquid theory

We will now review the quantities that are commonly calculated to describe many-body effects in solids, in order to highlight the microscopic foundations of the Fermi Liquid theory. As we have seen from a phenomenological perspective, a wide array of interacting solids exhibit excitations that resemble independent particles. Therefore, our first order of business is to define a physical quantity, called the Green's function, that fully characterizes the one-particle elementary excitations of a solid and that helps us discern if these resemble independent particle.

In order to do so, we consider a system of N electrons in its ground state $|\Psi_0\rangle$. At time $t = 0$ we inject a $(N + 1)$ electron with momentum k (one-particle excitation). The wavefunction that describes the system at time $t = 0^+$ is (in the occupation number representation, also known as second quantization)

$$|\Psi(t = 0)\rangle = a_k^\dagger |\Psi_0\rangle \quad (22)$$

Here a_k^\dagger (a_k) is the creation (annihilation) operator.

At time t the wavefunction will evolve through the action of the operator e^{-iHt}

$$|\Psi_1(t)\rangle = e^{-iHt} a_k^\dagger |\Psi_0\rangle \quad (23)$$

Now, let us consider a wavefunction at time t that describes the system of N particles and one additional electron that *has just been injected*. This wavefunction is

$$|\Psi_2(t)\rangle = a_k^\dagger e^{-iHt} |\Psi_0\rangle \quad (24)$$

We would like now to compare these two quantum states. *Our aim is to assess if the system with $N + 1$ particles that has had a time t to interact, still resembles a system with $N + 1$ particles with the additional particle that has just been injected.* A natural way to do this is to take the scalar product of the two wavefunctions we just considered

$$\langle \Psi_2(t) | \Psi_1(t) \rangle = \langle \Psi_0 | e^{iHt} a_k e^{-iHt} a_k^\dagger | \Psi_0 \rangle = \langle \Psi_0 | a_k(t) a_k^\dagger(0) | \Psi_0 \rangle \quad (25)$$

We take this as a definition for the single-particle correlation function $G(k, t)$ (or Green's function, valid only for $t > 0$)

$$iG(k, t) = \langle \Psi_0 | a_k(t) a_k^\dagger(0) | \Psi_0 \rangle \quad t > 0 \quad (26)$$

4.1 Non-interacting Green's function and spectral function

Let us assume that the electron that we have added is stable. It is an eigenstate of the non-interacting Hamiltonian

$$H = \sum_k \xi(k) a_k^\dagger a_k \quad (27)$$

(here $\xi(k)$ is the electron dispersion, or band structure, after subtraction of the chemical potential) and it does not interact with other degrees of freedom in the solid. In this case, the scalar product will be a simple phase factor with unit amplitude. The acquired phase is a function of the energy of this eigenstate. This is a description of the coherent evolution of the wavefunction that has not mixed with other excited states of the many-body system. Consequently, the Green's function will take the non-interacting form $G^0(k, t)$

$$G^0(k, t) = -i\theta(t)e^{-i\xi(k)t} \quad (28)$$

$\theta(t)$ is the Heaviside function defined as

$$\theta(t) = 0 \quad t < 0 \quad (29)$$

$$\theta(t) = 1 \quad t \geq 0 \quad (30)$$

In the frequency (energy) domain $G^0(k, \omega)$ takes the form

$$G^0(k, \omega) = \int dt e^{i\omega t} G^0(k, t) \quad (31)$$

$$G^0(k, \omega) = \frac{1}{\omega - \xi(k) + i0^+} \quad (32)$$

This result stems from the Fourier transform of the Heaviside function

$$\int dt e^{i\omega t} \theta(\pm t) = \frac{\pm i}{\omega \pm i0^+} \quad (33)$$

By inspecting equation 32, we realize that the Green's function exhibits poles at energies corresponding to one-particle excitations matching the dispersion relation.

At this point it is useful to define the spectral function

$$A(k, \omega) = -\frac{1}{\pi} \text{Im}G(k, \omega) \quad (34)$$

Remembering the Cauchy relation

$$\frac{1}{x \pm i0^+} = \text{pv} \frac{1}{x} \mp i\pi\delta(x) \quad (35)$$

we see that, for non-interacting electrons, the spectral function takes the form

$$A^0(k, \omega) = -\frac{1}{\pi} \text{Im}G^0(k, \omega) = \delta(\omega - \xi(k)) \quad (36)$$

The spectral function exhibits sharp peaks at energies corresponding to one-particle excitations matching the dispersion relation. As we will see later, this *quantity can be measured directly in experiments*. We interpret this quantity as a probability to find a one-particle excitation with frequency ω and momentum k . Therefore we expect the following relation to hold (known as sum rule)

$$\int_{-\infty}^{+\infty} A(k, \omega) d\omega = 1 \quad (37)$$

4.2 Interacting Green's function and spectral function

In the presence of interactions between particles, the evolution operator will bring about a decomposition on the spectrum of excited states of the system. As a result the scalar product will not be a mere phase factor with constant amplitude. Rather, its amplitude will decrease as a function of time. A characteristic form of this decrease is an exponential decay. This describes the loss of coherence of the wave function brought about by interactions. After a characteristic time (lifetime), the one-particle excitation loses any resemblance to the free electron. The most simple form of interacting Green's function is

$$G(k, t) = -i\theta(t)e^{-i\xi(k)t}e^{-t/\tau} \quad (38)$$

to be compared with the non-interacting case

$$G^0(k, t) = -i\theta(t)e^{-i\xi(k)t} \quad (39)$$

The spectral function gauges the weight of each excited state of the decomposition and takes in this simple case the form of a Lorentzian peak

$$A(k, \omega) = \frac{1}{\pi} \frac{1/\tau}{(\omega - \xi(k))^2 + (1/\tau)^2} \quad (40)$$

to be compared with the non-interacting case

$$A^0(k, \omega) = \delta(\omega - \xi(k)) \quad (41)$$

In more general terms, the Green's function for an interacting problem can be written as

$$G(k, \omega) = \frac{1}{\omega - \xi(k) - \Sigma(k, \omega) + i0^+} \quad (42)$$

$\Sigma(k, \omega)$ is called the self-energy. It is, in general, a function of momentum and frequency. Its computation is a central problem in many-body theory as it accounts for the effects of interactions.

The corresponding spectral function is

$$A(k, \omega) = -\frac{1}{\pi} \frac{\text{Im}\Sigma(k, \omega)}{(\omega - \xi(k) - \text{Re}\Sigma(k, \omega))^2 + (\text{Im}\Sigma(k, \omega))^2} \quad (43)$$

The self-energy contains all the information necessary in order to characterize the effect of interactions on the single-particle excitation spectrum. It is in general a complex quantity, with a real and an imaginary part and a dependence on momentum and frequency.

We will now describe the information contained in the real and imaginary part of the self energy separately.

4.3 Imaginary part of the self-energy: lifetime

We will now assume that the real part of the self-energy is zero in order to highlight the information contained in its imaginary part. Moreover, we will assume that the dependence of this quantity on momentum and frequency is sufficiently weak that we can consider it as a constant. In this approximation the spectral function reads:

$$A(k, \omega) = -\frac{1}{\pi} \frac{\text{Im}\Sigma}{(\omega - \xi(k))^2 + (\text{Im}\Sigma)^2} \quad (44)$$

By comparing this form with the simplest form of spectral function for the interacting problem

$$A(k, \omega) = \frac{1}{\pi} \frac{1/\tau}{(\omega - \xi(k))^2 + (1/\tau)^2} \quad (45)$$

we identify the imaginary part of the self-energy as a generalization of the lifetime

$$1/\tau = -\text{Im}\Sigma \quad (46)$$

4.4 Real part of the self-energy: effective mass and quasiparticle residue

We will now assume that the imaginary part of the self-energy is zero, in order to highlight the information contained in its real part. In this approximation, the spectral function reads

$$A(k, \omega) = \delta(\omega - \xi(k) - \text{Re}\Sigma(k, \omega)) \quad (47)$$

From this equation, we understand that the effect of the real part of the self-energy is to shift the position of the peak. This affects directly the dispersion relation of the interacting electrons and therefore has significant physical consequences. The new dispersion relation $\epsilon(k)$ is defined as

$$\epsilon(k) = \xi(k) + \text{Re}\Sigma(k, \omega = \epsilon(k)) \quad (48)$$

In order to proceed with the calculation we will assume the simplest form of non-interacting dispersion relation: isotropic and well approximated by the one of free electrons ($m = m_e = 9.1 \cdot 10^{-31}$ Kg).

$$\xi(k) = \frac{k^2 - k_F^2}{2m} \quad (49)$$

In general m can already be affected by the band structure of the material.

Since we are only interested in excitations close to the Fermi energy, it is useful to approximate the dispersion with a linear form. A first order Taylor expansion of the free dispersion around the Fermi energy reads

$$\xi(k) \simeq \frac{k_F}{m}(k - k_F) \quad (50)$$

The same dispersion for the renormalized dispersion is

$$\epsilon(k) \simeq \frac{k_F}{m^*}(k - k_F) \quad (51)$$

We are now in a position to find a quantitative relation between the real part of the self-energy and the renormalisation of the electron mass m/m^* . In order to do so, it is useful to compute $d\epsilon(k)/dk$.

$$\frac{d\epsilon(k)}{dk} = \frac{d\xi(k)}{dk} + \left. \frac{\partial \text{Re}\Sigma(k, \omega)}{\partial k} \right|_{\omega=\epsilon(k)} + \left. \frac{\partial \text{Re}\Sigma(k, \omega)}{\partial \omega} \right|_{\omega=\epsilon(k)} \frac{d\epsilon(k)}{dk} \quad (52)$$

By direct computation we find

$$\frac{d\xi(k)}{dk} = \frac{k_F}{m} \quad (53)$$

and

$$\frac{d\epsilon(k)}{dk} = \frac{k_F}{m^*} \quad (54)$$

Substituting these two expressions in equation 52, we obtain the relation

$$\frac{k_F}{m^*} = \frac{k_F}{m} + \left. \frac{\partial \text{Re}\Sigma(k, \omega)}{\partial k} \right|_{\omega=\epsilon(k)} + \frac{k_F}{m^*} \left. \frac{\partial \text{Re}\Sigma(k, \omega)}{\partial \omega} \right|_{\omega=\epsilon(k)} \quad (55)$$

which can be used to compute m/m^*

$$\frac{m}{m^*} = \frac{1 + \frac{m}{k_F} \left. \frac{\partial \text{Re}\Sigma(k, \omega)}{\partial k} \right|_{\omega=\epsilon(k)}}{1 - \left. \frac{\partial \text{Re}\Sigma(k, \omega)}{\partial \omega} \right|_{\omega=\epsilon(k)}} \quad (56)$$

This is an important result as it gives us a microscopic mechanism for understanding why many interacting quantum particles exhibit thermodynamic responses that resembles the

ones of free electrons with vastly different masses. This argument forms the microscopic basis for supporting a Fermi liquids description of metals.

There is a second important effect generated by the real part of the self-energy. In order to highlight it, we need to express the spectral function in terms of the renormalized dispersion. This can be obtained by using the following relation for the δ distribution

$$\delta(\phi(x)) = \sum_i \frac{1}{|\phi'(x_i)|} \delta(x - x_i) \quad \text{for } \phi(x_i) = 0 \quad (57)$$

Applying this formula we obtain

$$A(k, \omega) = \delta(\omega - \xi(k) - \text{Re}\Sigma(k, \omega)) = \quad (58)$$

$$= \frac{1}{\left. \frac{\partial}{\partial \omega} (\omega - \xi(k) - \text{Re}\Sigma(k, \omega)) \right|_{\omega=\epsilon(k)}} \delta(\omega - \epsilon(k)) = Z_k \delta(\omega - \epsilon(k)) \quad (59)$$

Remarkably, instead of obtaining a relation for the spectral function of the form $A \sim \delta(\omega - \epsilon(k))$ we have found

$$A(k, \omega) = Z_k \delta(\omega - \epsilon(k)) \quad (60)$$

The quantity

$$Z_k = \frac{1}{\left. 1 - \frac{\partial \text{Re}\Sigma(k, \omega)}{\partial \omega} \right|_{\omega=\epsilon(k)}} \quad (61)$$

is known as the quasiparticle residue. Remembering equation 56, we find the following relation valid in the limit in which the momentum dependence of the real part of the self-energy is small compared to the energy dependence (which in practice is relevant for many situation).

$$\frac{m}{m^*} \simeq Z_k \quad (62)$$

$$m^* = \left(1 - \left. \frac{\partial \text{Re}\Sigma}{\partial \omega} \right|_{\omega=\epsilon(k)} \right) m \quad (63)$$

We previously introduced the sum rule (equation 37). Z_k is in general a number smaller than 1. Therefore an energy dependence of the real part of the self-energy causes the spectral weight of an excitation to be less than 1. This is an important realization: interactions have transformed only part of the spectral weight of an electron into a new excitation with a different dispersion relation and a heavier mass.

The results of our calculations are summarized in figure 4

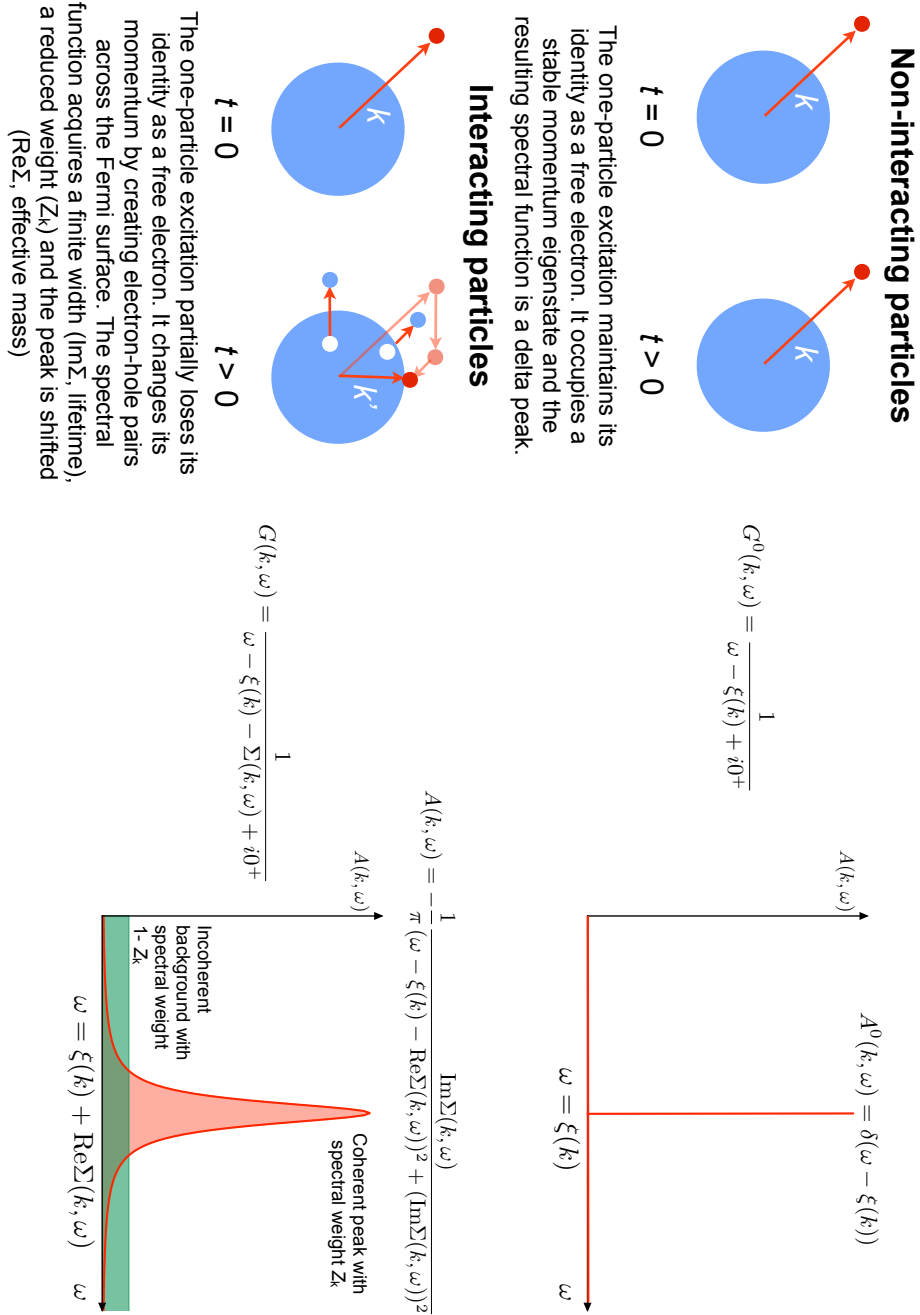


Figure 4: Summary of the quantities used in order to describe many-body effects in solids