Lecture notes on the BCS theory of superconductivity

Contents

1 Effective electron-electron interactions

During lectures 6 and 9, we have introduced the concept that elementary interactions in solids can acquire a non-trivial dependence on temporal (ω) and spatial (k) variables (renormalisation of the interactions), that accounts for many-body effects and the emergence of collective modes. A notable example is the phenomenon of screening, that comes into play in the description of electron-electron interactions and phonon frequencies in metals. The potential of a charge Q is screened by the conduction electrons of a metal, according to the Thomas-Fermi theory (also known as random phase approximation, discussed in lecture 6), as

$$V(r) = \frac{Q}{r} e^{-k_{\mathrm{TF}} r} \tag{1}$$

where $\lambda_{\mathrm{TF}} = 1/k_{\mathrm{TF}}$ is the Thomas-Fermi screening length, of the order 0.5 to 1 nm in typical metals. The bare electron-electron interaction

$$V_{\mathrm{bare}}(r) = \frac{e}{4\pi\epsilon_0} \frac{1}{r} \tag{2}$$

that has a momentum dependence

$$V_{\mathrm{bare}}(k) = \frac{e}{4\pi\epsilon_0} \frac{1}{k^2} \tag{3}$$

becomes dressed by interactions as

$$V_{\mathrm{dressed}}(\mathbf{k}, \omega) = \frac{e}{4\pi\epsilon_0 \epsilon(\mathbf{k}, \omega)} \frac{1}{k^2} = \frac{V_{\mathrm{bare}}(k)}{\epsilon(\mathbf{k}, \omega)} \tag{4}$$

as described by the dielectric function $\epsilon(\mathbf{k}, \omega)$.

In the Thomas-Fermi approximation, screening yields the Yukawa potential

$$V^{\mathrm{Y}}(\mathbf{k}, 0) = \frac{e}{4\pi\epsilon_0} \frac{1}{k^2 + k_{\mathrm{TF}}^2} \tag{5}$$

where it is assumed that screening by conduction electrons occurs instantaneously. In this case the dielectric function shows no ω dependence:

$$\frac{1}{\epsilon^{el}(\mathbf{k}, 0)} = \frac{1}{1 + \frac{k_{\mathrm{TF}}^2}{k^2}} = \frac{k^2}{k^2 + k_{\mathrm{TF}}^2} \tag{6}$$

That can be also written as

$$\epsilon^{el}(\mathbf{k}, 0) = 1 + \frac{k_{\mathrm{TF}}^2}{k^2} \tag{7}$$

Here we want to go beyond this approximation and include also the effects associated with the dynamics of the crystal lattice (phonons). We approximate the lattice dynamics

as a collective excitation of positively charged ions (jellium model). In this approximation we calculate the phonon frequencies of metals starting from their bare plasma frequency

$$\omega_{\text{plasma ions}} = \frac{n_i (Ze)^2}{M_i \epsilon_0} \tag{8}$$

where n_i, M_i, Z are the ion density, mass and charge respectively. The presence of the conduction electrons in a metal, renormalises the Coulomb potential and hence the restoring force of an ionic lattice, leading to a phonon dispersion $\omega_{\mathbf{k}}$

$$\omega_{\mathbf{k}}^2 = \frac{\omega_{\text{plasma ions}}^2}{\epsilon^{el}(\mathbf{k}, 0)} = \frac{\omega_{\text{plasma ions}}^2}{k^2 + k_{\text{TF}}^2} k^2 \tag{9}$$

This can be approximated at small momenta as

$$\omega_{\mathbf{k}} \simeq \frac{\omega_{\text{plasma ions}}}{k_{\text{TF}}} k = v_{\text{sound}} k \tag{10}$$

The dielectric function for the ionic lattice can be well approximated as a momentum-independent quantity, with a temporal dependence representing the plasma response

$$\epsilon^{ion}(0, \omega) = 1 - \frac{\omega_{\text{plasma ions}}^2}{\omega^2} \tag{11}$$

This quantity accounts for the finite response time of a lattice polarisation, typically bounded by the Debye frequency (in the 100 fs range).

We now want to compute a general form for the renormalisation of an electron-electron interaction potential in the presence of 1) polarisation of the ionic potential and its retarted screening and 2) instantaneous and momentum dependent electronic screening. As introduced above, the renormalised potential is related to the bare potential by the total dielectric function

$$V_{\text{dressed}}(\mathbf{k}, \omega) = V_{\text{bare}}(k) / \epsilon(\mathbf{k}, \omega) \tag{12}$$

The electronic system screens the combined action of the bare potential and ionic potential

$$V_{\text{dressed}}(\mathbf{k}, \omega) = (V_{\text{bare}} + V_{\text{ion}}) / \epsilon^{el}(\mathbf{k}, 0) \tag{13}$$

while the ion system screens the bare potential supplemented by the electron polarisation

$$V_{\text{dressed}}(\mathbf{k}, \omega) = (V_{\text{bare}} + V_{\text{el}}) / \epsilon^{ion}(0, \omega) \tag{14}$$

By adding the last two equations and substracting the previous one, we come to

$$V_{\text{dressed}} = V_{\text{bare}} + V_{\text{el}} + V_{\text{ion}} = \left[\epsilon^{el}(\mathbf{k}, 0) + \epsilon^{ion}(0, \omega) - \epsilon(\mathbf{k}, \omega) \right] V_{\text{dressed}} \tag{15}$$

which implies

$$\epsilon(\mathbf{k},\omega) = \epsilon^{el}(\mathbf{k},0) + \epsilon^{ion}(0,\omega) - 1 \tag{16}$$

By substituting we find the total dielectric function

$$\epsilon(\mathbf{k},\omega) = 1 + \frac{k_{\mathrm{TF}}^2}{k^2} - \frac{\omega^2_{\text{plasma ions}}}{\omega^2} = \left(1 + \frac{k_{\mathrm{TF}}^2}{k^2}\right)\left(1 - \frac{\omega_{\mathbf{k}}^2}{\omega^2}\right) \tag{17}$$

This allows us to write the dressed electron-electron interaction as

$$V_{\text{dressed}}(\mathbf{k},\omega) = \frac{e}{4\pi\epsilon_0\epsilon(\mathbf{k},\omega)}\frac{1}{k^2} = \frac{e}{4\pi\epsilon_0}\frac{1}{k_{\mathrm{TF}}^2 + k^2}\left(1 + \frac{\omega_{\mathbf{k}}^2}{\omega^2 - \omega_{\mathbf{k}}^2}\right) \tag{18}$$

or

$$V_{\text{dressed}}(\mathbf{k},\omega) = \frac{e}{4\pi\epsilon_0}\frac{1}{k_{\mathrm{TF}}^2 + k^2} + \frac{e}{4\pi\epsilon_0}\frac{1}{k_{\mathrm{TF}}^2 + k^2}\frac{\omega_{\mathbf{k}}^2}{\omega^2 - \omega_{\mathbf{k}}^2} \tag{19}$$

The first term represents the renormalised Coulomb repulsion while the second term represents an electron-phonon mediated interaction. The second term involves the dynamics of phonons with $\omega_{\mathbf{k}}$ describing the phonon dispersion, bounded by the Debye frequency ω_D.

Importantly, we have found that, for frequencies below the cut-off of the phonon spectrum ω_D, the interaction is attractive. In this regime the polarisation of the lattice can be described as an oscillator excited below its resonance, therefore oscillating in-phase with the electron and providing a singular attractive contribution at the lattice resonance ω_k. For frequencies above the characterisitic phonon frequency (Debye), the effective interaction is repulsive. The lattice is driven above its resonance and oscillates out of phase with the electron, making the electron repulsion even more effective than the bare Coulomb interaction.

Based on this result, we express the effective electron-electron interaction as a two-body potential

$$H_{\text{int}} = \frac{1}{2\mathscr{V}}\sum_{\mathbf{k}\mathbf{k}'}\sum_{\sigma,\sigma\prime} V_{\mathbf{k}\mathbf{k}'} c^\dagger_{\mathbf{k}\sigma} c^\dagger_{-\mathbf{k}\sigma\prime} c_{-\mathbf{k}'\sigma\prime} c_{\mathbf{k}'\sigma} \tag{20}$$

with

$$V_{\mathbf{k}\mathbf{k}'} = \frac{e}{4\pi\epsilon_0}\frac{1}{q_{\mathrm{TF}}^2 + q^2} + \frac{e}{4\pi\epsilon_0}\frac{1}{q_{\mathrm{TF}}^2 + q^2}\frac{\omega_{\mathbf{q}}^2}{\omega^2 - \omega_{\mathbf{q}}^2} \tag{21}$$

and $\mathbf{q} = \mathbf{k} - \mathbf{k}'$

2 Instability of the Fermi liquid

In this section we want to introduce a very important concept: the instability of the Fermi liquid in the presence of an attractive interaction between electrons (Cooper instability). We have seen previously that a Fermi liquid is a state of matter that is smoothly connected

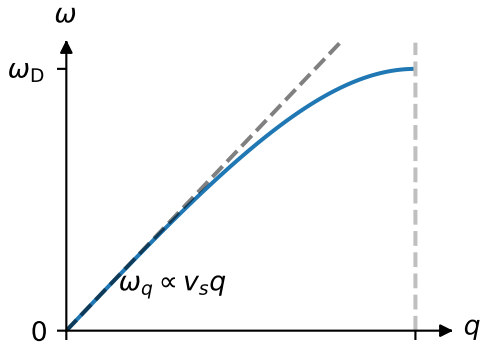

Figure 1: Phonon dispersion limited by the Debye frequency.

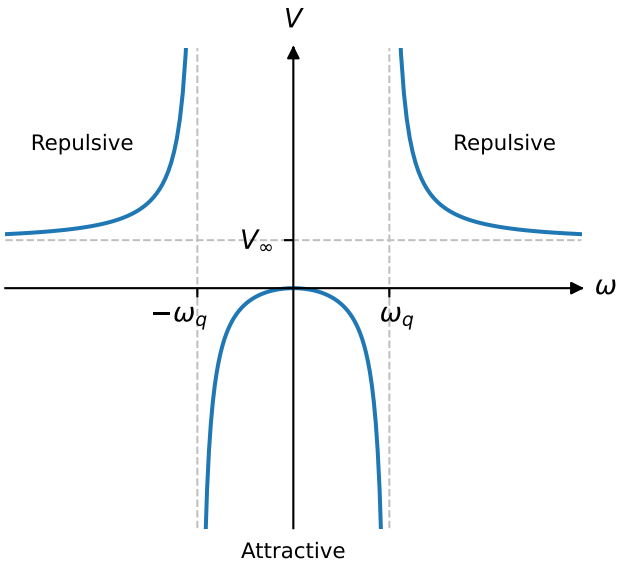

Figure 2: Effective electron-electron interaction

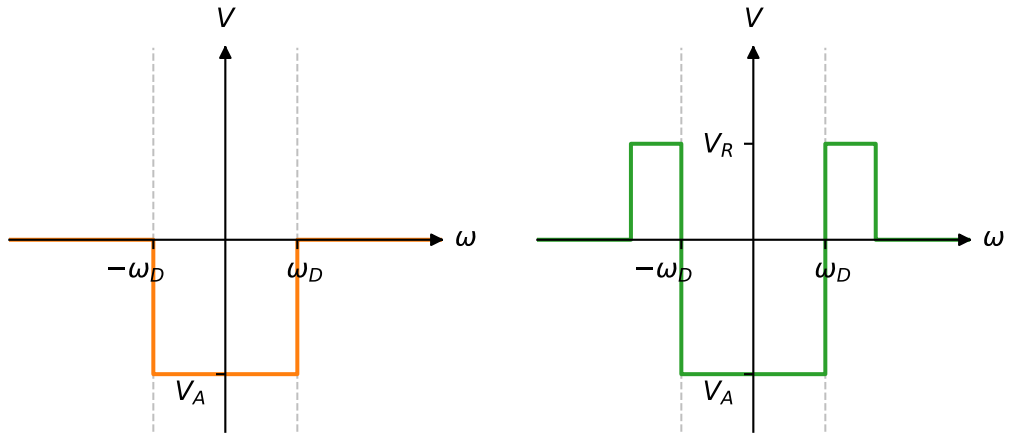

Figure 3: One-square well and two-square well models

to the free Fermi gas by perturbation theory. We are able to define a perturbative expansion in powers of the interaction that connects wave functions and eigenenergies of the Fermi gas to those of an interacting Fermi liquid.

In contrast, we can find many-body systems with scattering mechanisms that lead to divergent perturbative expansions in the interaction strength V starting from the Fermi gas. They can be responsible for an instability of the (quasi) free electron gas and a phase transition towards an entirely different state of matter.

Our first step will be to demonstrate that, if there is a scattering mechanism that leads to an effective attractive interaction between pairs of electrons V, then, no matter how small this attractive interaction might be, the Fermi liquid is not the ground state of the electron system. Another state of matter is characterised by a lower energy Δ.

We will first restrict ourselves to the case of a two-electron wavefunction and compute a relation $\Delta(V)$ by solving the Schroedinger equation. If we can show that $\Delta > 0$, we will have uncovered an instability of the Fermi liquid (there is a paired state that is always energetically favourable compared to the unpaired configuration). We will then consider the case of a many-particle system by diagonalising the Bardeen, Cooper and Schrieffer (BCS) hamiltonian in the mean field approximation using a canonical transformation. This analysis will allow us to account for the ground state and excited states of a superconductor in the weak coupling limit ($g(E_F)V \ll 1$).

The following scenario was originally formulated by Cooper (Physical Review 104, 1189 (1956)) and it is known as the Cooper problem. Let's consider two electrons located just outside the Fermi sphere, not interacting with the other electrons, banished from the inside of the Fermi sphere by the Pauli exclusion principle but interacting with each other through a potential $V(\mathbf{r}_1, \mathbf{r}_2)$. The Schroedinger equation for their two-electron wave function

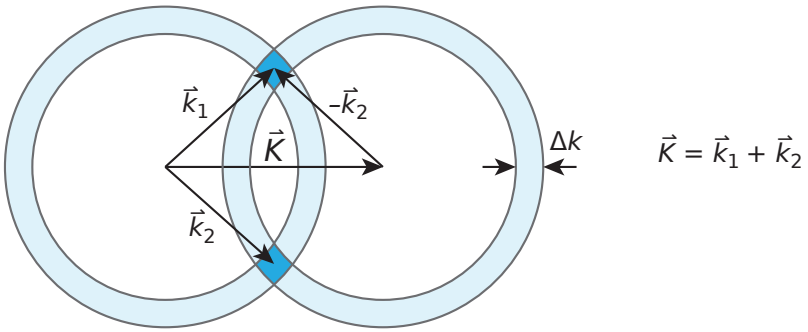

Figure 4: Phase space for interacting electrons above the Fermi surface

$\psi(\mathbf{r}_1, \mathbf{r}_2)$ reads

$$-\frac{\hbar^2}{2m}\left(\nabla_1^2 + \nabla_2^2\right)\psi(\mathbf{r}_1, \mathbf{r}_2) + V(\mathbf{r}_1, \mathbf{r}_2)\psi(\mathbf{r}_1, \mathbf{r}_2) = \epsilon\psi(\mathbf{r}_1, \mathbf{r}_2) \tag{22}$$

According to our definition of the change in ground state energy outlined above (Δ), it is instructive to rewrite it as

$$-\frac{\hbar^2}{2m}\left(\nabla_1^2 + \nabla_2^2\right)\psi(\mathbf{r}_1, \mathbf{r}_2) + V(\mathbf{r}_1, \mathbf{r}_2)\psi(\mathbf{r}_1, \mathbf{r}_2) = \epsilon\psi(\mathbf{r}_1, \mathbf{r}_2) = (2E_F - \Delta)\psi(\mathbf{r}_1, \mathbf{r}_2) \tag{23}$$

where Δ is the energy gain obtained from the pair formation with respect to the energy without interaction, in which each electron has energy of the order E_F.

We will show that for an arbitrarily weak attractive interaction the two electrons will form a bound state ($\Delta > 0$). In the section above, we have discussed the nature of the effective electron-electron interaction in metals, that shows an attractive behaviour for energies below the characteristic Debye frequency ω_D. The two electrons interact by exchanging phonons, therefore energy conservation limits the possible phase space available for interactions to a spherical shell between E_F and $E_F + \hbar\omega_D$. We also know that their total momentum must be conserved

$$\mathbf{K} = \mathbf{k}_1 + \mathbf{k}_2 \tag{24}$$

limiting further the phase space to the intersection between two spherical shells as illustrated in the figure above. The volume of this phase space is maximised for $\mathbf{K} = 0$, therefore we restrict ourselves to interacting electrons with opposite momenta[1].

$$\mathbf{k}_1 = -\mathbf{k}_2 = \mathbf{k} \tag{25}$$

[1]Note that we cannot exclude a priori the possibility of finite-momentum pairing. There are examples of exotic superconductors where this scenario is considered.

For $V = 0$ (non interacting case) we have

$$\psi_0(\mathbf{r}_1, \mathbf{r}_2) = \left(\frac{1}{\sqrt{\mathscr{V}}} e^{i\mathbf{k}_1 \cdot \mathbf{r}_1}\right)\left(\frac{1}{\sqrt{\mathscr{V}}} e^{i\mathbf{k}_2 \cdot \mathbf{r}_2}\right) = \frac{1}{\mathscr{V}} e^{i\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)} = \psi_0(\mathbf{r}_1 - \mathbf{r}_2) \tag{26}$$

We note that the spatial part of the wave function is symmetric with respect to the exchange of particle. In order for the total wave function to be antisymmetric with respect to the exchange of particle, the spin part must be a singlet.

$$\chi_a = \frac{1}{\sqrt{2}} \left(|\uparrow_1, \downarrow_2\rangle - |\downarrow_1, \uparrow_2\rangle\right) \tag{27}$$

For $V \neq 0$ (interacting case) we can expand the two-electron wave function as

$$\psi(\mathbf{r}_1 - \mathbf{r}_2) = \frac{1}{\mathscr{V}} \sum_{\mathbf{k}} C(\mathbf{k}) e^{i\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)} \tag{28}$$

Since we are dealing with attractive interactions, the lowest energy solution will be expanded by terms of the kind $\cos[\mathbf{k} \cdot (\mathbf{r}_1 - \mathbf{r}_2)]$ (having maximum amplitude for bonding states). These are spatially symmetric wave functions, therefore the above consideration on the spin of the pair still holds.

Without loss of generality we will use the coordinate $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

$$\psi(\mathbf{r}) = \frac{1}{\mathscr{V}} \sum_{\mathbf{k}} C(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} \tag{29}$$

Plugging these expansions into the Schroedinger equation we obtain

$$-\frac{\hbar^2}{2m}\nabla_1^2 \sum_{\mathbf{k}} C(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} - \frac{\hbar^2}{2m}\nabla_2^2 \sum_{\mathbf{k}} C(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} + V(\mathbf{r}) \sum_{\mathbf{k}} C(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} = (2E_F - \Delta) \sum_{\mathbf{k}} C(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} \tag{30}$$

$$\frac{\hbar^2}{m} \sum_{\mathbf{k}} C(\mathbf{k}) k^2 e^{i\mathbf{k}\cdot\mathbf{r}} + V(\mathbf{r}) \sum_{\mathbf{k}} C(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} = (2E_F - \Delta) \sum_{\mathbf{k}} C(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} \tag{31}$$

We now make use of the Fourier transform $\int_{\mathscr{V}} e^{-i\mathbf{k}'\cdot\mathbf{r}} d\mathbf{r}$

$$\frac{\hbar^2}{m} \sum_{\mathbf{k}} C(\mathbf{k}) k^2 \int_{\mathscr{V}} e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} d\mathbf{r} + \sum_{\mathbf{k}} C(\mathbf{k}) \int_{\mathscr{V}} V(\mathbf{r}) e^{-i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}} d\mathbf{r} = (2E_F - \Delta) \sum_{\mathbf{k}} C(\mathbf{k}) \int_{\mathscr{V}} e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} d\mathbf{r} \tag{32}$$

We know the following relation for the Fourier transform

$$\int_{\mathscr{V}} d\mathbf{r} e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} = \mathscr{V}\delta(\mathbf{k} - \mathbf{k}') = \mathscr{V}\delta_{\mathbf{k}\mathbf{k}'} \tag{33}$$

which leads to

$$\frac{\hbar^2}{m}\sum_{\mathbf{k}} C(\mathbf{k})k^2\mathscr{V}\delta_{\mathbf{k}\mathbf{k}'} + \sum_{\mathbf{k}} C(\mathbf{k})V(\mathbf{k}'-\mathbf{k}) = (2E_F-\Delta)\sum_{\mathbf{k}} C(\mathbf{k})\mathscr{V}\delta_{\mathbf{k}\mathbf{k}'} \tag{34}$$

where $V(\mathbf{k}-\mathbf{k}') = V_{\mathbf{k}\mathbf{k}'}$ is the Fourier transform (or matrix elements) of the interaction potential

$$\frac{\hbar^2\mathbf{k}'^2}{m}\mathscr{V}C(\mathbf{k}') + \sum_{\mathbf{k}} C(\mathbf{k})V(\mathbf{k}'-\mathbf{k}) = (2E_F-\Delta)\mathscr{V}C(\mathbf{k}') \tag{35}$$

We are free to exchange the indexes as

$$\frac{\hbar^2\mathbf{k}^2}{m}C(\mathbf{k}) + \frac{1}{\mathscr{V}}\sum_{\mathbf{k}'} C(\mathbf{k}')V(\mathbf{k}-\mathbf{k}') = (2E_F-\Delta)C(\mathbf{k}) \tag{36}$$

We need now to specify a form for the interaction potential. We have calculated an approximated form of effective $V_{\mathbf{k}\mathbf{k}'}$ in the previous section which is in principle correct but hard to treat analytically. Cooper proposed to use as a simple approximation for the interaction a constant, k-independent attraction active only within the top spherical shell bounded by the Debye frequency (see see figure 3, Cooper, Phys. Rev. 104, 1189 (1956)).

$$V_{\mathbf{k}\mathbf{k}'} = -V < 0 \text{ for } k_F < [k,k'] < \sqrt{\frac{2m(E_F+\hbar\omega_D)}{\hbar^2}} \tag{37}$$

$$V_{\mathbf{k}\mathbf{k}'} = 0 \text{ everywhere else} \tag{38}$$

where $V > 0$.

We define as $\mathbf{k}_{\text{shell}}$ the set of wavevectors k such as $k_F < k < \sqrt{\frac{2m(E_F+\hbar\omega_D)}{\hbar^2}}$. Thanks to this hypothesis on the form of the interaction and to the Pauli principle we can restrict the expansion $C(\mathbf{k})$ to $\mathbf{k} \in \mathbf{k}_{\text{shell}}$ and impose

$$C(\mathbf{k}) = 0 \text{ for } k < k_F \tag{39}$$

and

$$C(\mathbf{k}) = 0 \text{ for } k > \sqrt{\frac{2m(E_F+\hbar\omega_D)}{\hbar^2}} \tag{40}$$

Accordingly, we will limit the summation as

$$\frac{\hbar^2\mathbf{k}^2}{m}C(\mathbf{k}) - \frac{V}{\mathscr{V}}\sum_{\mathbf{k}'\in\mathbf{k}_{\text{shell}}} C(\mathbf{k}') = (2E_F-\Delta)C(\mathbf{k}) \tag{41}$$

$A = \frac{V}{\mathscr{V}}\sum_{\mathbf{k}'\in\mathbf{k}_{\text{shell}}} C(\mathbf{k}')$ is a constant number independent on $\mathbf{k}$

$$\left(-\frac{\hbar^2\mathbf{k}^2}{m} - \Delta + 2E_F\right)C(\mathbf{k}) = -A \tag{42}$$

We are not interested in an explicit calculation of the wavefunctions, rather to gain insight on the relation between eigenenergies and interaction strength. Therefore we will sum up the remaining k dependence of the expansions and aim to isolate a form $\Delta(V)$.

$$\sum_{\mathbf{k}} C(\mathbf{k}) = A \sum_{\mathbf{k}} \frac{1}{\frac{\hbar^2 \mathbf{k}^2}{m} - \Delta + 2E_F} \tag{43}$$

$$1 = \frac{V}{\mathscr{V}} \sum_{\mathbf{k}} \frac{1}{\frac{\hbar^2 \mathbf{k}^2}{m} - \Delta + 2E_F} \tag{44}$$

Since the sum only depends on $\mathbf{k}^2$, we can easily transform the discrete k summation into an energy integral using the density of states $g(\epsilon)$ and $\epsilon = \frac{\hbar^2 k^2}{2m}$

$$\frac{1}{\mathscr{V}} \sum_{\mathbf{k}} \ ... \ \rightarrow \underbrace{2}_{\text{spin degeneracy}} \int \ ... \ g(\epsilon) d\epsilon \tag{45}$$

resulting in

$$1 = 2V \int_{E_F}^{E_F + \hbar\omega_D} \frac{g(\epsilon) d\epsilon}{2\epsilon + \Delta - 2E_F} \tag{46}$$

In most conventional metals we can safely assume that $g(\epsilon)$ varies weakly around E_F, allowing us to perform the variable change $\eta = \epsilon - E_F$ and reduce the integral to

$$1 = g(E_F) V \left(2 \int_0^{\hbar\omega_D} \frac{d\eta}{2\eta + \Delta} \right) \tag{47}$$

$$2 \int_0^{\hbar\omega_D} \frac{d\eta}{2\eta + \Delta} = 2 \frac{1}{2} \int_0^{2\hbar\omega_D} \frac{d(2\eta)}{2\eta + \Delta} = \ln \frac{2\hbar\omega_D + \Delta}{0 + \Delta} = \ln \left(1 + \frac{2\hbar\omega_D}{\Delta} \right) \tag{48}$$

We can now solve for $\Delta(V)$

$$\frac{1}{g(E_F)V} = \ln \left(1 + \frac{2\hbar\omega_D}{\Delta} \right) \tag{49}$$

$$e^{\frac{1}{g(E_F)V}} = 1 + \frac{2\hbar\omega_D}{\Delta} \tag{50}$$

$$\Delta = \frac{2\hbar\omega_D}{e^{\frac{1}{g(E_F)V}} - 1} \tag{51}$$

In the weak coupling limit (coupling constant $\lambda = g(E_F)V \ll 1$, valid for elemental superconductors) we find

$$\boxed{\Delta \simeq 2\hbar\omega_D e^{-\frac{1}{g(E_F)V}} = 2\hbar\omega_D e^{-\frac{1}{\lambda}} > 0} \tag{52}$$

We have demonstrated that, for an arbitrarily small and crude attractive interaction, two electrons living at the edge of the Fermi sphere will always form a bound state. We call this bound state a Cooper pair. We can reasonably extend our pairing interaction to all other electrons at the edge of the Fermi sphere and argue that in the presence of an arbitrarily weak two-body attractive intreraction at the edge of the Fermi sphere, the Fermi gas is not the ground state of a metal. This instability will lead to the formation of a great number of electron pairs with opposite momenta and singlet spins, that form bound states gaining an energy Δ, known as energy gap.

The characteristic spatial size of Cooper pairs, the coherence length, can be estimated from the quantization of the action $\delta p \delta x \simeq \hbar$ (see exercises). We know that pairs are formed by electrons in a layer $\pm\Delta$ around the edge of the Fermi sphere, so we find characteristic size of the Cooper pair is

$$\delta x \simeq \frac{\hbar}{\delta p} \simeq \frac{\hbar v_F}{\Delta} = \frac{4 \cdot 10^{-15} \text{ eV s} \times 10^6 \text{ m s}^{-1}}{10^{-3} \text{ eV}} \sim 10^{-6} \div 10^{-7} \text{ m} \tag{53}$$

We note that this number is much larger than the crystalline unit cell: Cooper pairs should not be imagined as localised atom-like electron pairs. They are, rather, correlations between pairs of wavefunctions that extend over large distances (Al 1600 nm, Pb 80 nm, $SrTiO_3$ 70 nm).

As we will see below, this scenario based on a two-body pairing is our best current theoretical starting point for describing the superconducting state in the weak coupling limit. Importantly, we have seen that the energy gap Δ is not an analytic function of the interaction V (or the coupling constant λ), therefore the state of matter we are investigating cannot be connected to the Fermi gas (or liquid) by a perturbative expansion. This fact underscores the failure of perturbation theory to provide a description of the superconducting state and highlights the need for a different theoretical starting point. A mean field analysis will give us the necessary tools to make progress.

3 BCS mean-field hamiltonian

We wish to extend our analysis from a two-electron scenario to the case of a many-body system. We consider the hamiltonian with a two-body interaction:

$$H = \sum_{\mathbf{k}\sigma} \xi(\mathbf{k}) c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \frac{1}{2\mathscr{V}} \sum_{\mathbf{k}\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} \tag{54}$$

Since we have understood that perturbation theory is not an appropriate tool for the problem at hand, we need to elaborate a different approach for finding approximate solutions. In a mean-field approximation, we consider the operators A and B and their product AB. We rewrite the operators as their thermodynamic average $\langle\rangle$ plus a fluctuation term

$$A = \langle A \rangle + \delta A \tag{55}$$

$$B = \langle B \rangle + \delta B \tag{56}$$

$$AB = \langle A \rangle \langle B \rangle + \langle A \rangle \delta B + \langle B \rangle \delta A + \delta A \delta B \tag{57}$$

We assume that the fluctuations around the average are small and neglect the term quadratic in the fluctuations

$$\delta A \delta B \sim 0 \tag{58}$$

Replacing the fluctuations as $\delta A = A - \langle A \rangle$ and $\delta B = B - \langle B \rangle$ we come to the expression

$$AB = \langle A \rangle B + \langle B \rangle A - \langle A \rangle \langle B \rangle \tag{59}$$

In the BCS mean-field approximation we choose $A = c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}$ and $B = c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow}$ leading to

$$H^{\mathrm{BCS}} = \sum_{\mathbf{k}\sigma} \xi(\mathbf{k}) c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \tag{60}$$

$$\frac{1}{2\mathscr{V}} \sum_{\mathbf{k}\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \left(\langle c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} \rangle c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} + \langle c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} \rangle c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} - \langle c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} \rangle \langle c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} \rangle \right)$$

We can simplify this expression by denoting the average terms as

$$\Delta_{\mathbf{k}} = -\frac{1}{2\mathscr{V}} \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \langle c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} \rangle \tag{61}$$

$$\Delta^*_{\mathbf{k}} = -\frac{1}{2\mathscr{V}} \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \langle c^\dagger_{\mathbf{k}'\uparrow} c^\dagger_{-\mathbf{k}'\downarrow} \rangle \tag{62}$$

leading to

$$\begin{aligned} H^{\mathrm{BCS}} &= \sum_{\mathbf{k}\sigma} \xi(\mathbf{k}) c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - \sum_{\mathbf{k}} \Delta^*_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} - \sum_{\mathbf{k}} \Delta_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} \\ &= \sum_{\mathbf{k}} \left[\xi(\mathbf{k}) (c^\dagger_{\mathbf{k}\uparrow} c_{\mathbf{k}\uparrow} + c^\dagger_{\mathbf{k}\downarrow} c_{\mathbf{k}\downarrow}) - \Delta^*_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} - \Delta_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} \right] \end{aligned} \tag{63}$$

The last term $(-\Delta^*_{\mathbf{k}} \Delta_{\mathbf{k}} / V)$ has been omitted as being constant. Its role will be discussed separately. It is noteworthy that H^{BCS} does not conserve the number of particles. When expressed in this form, it becomes apparent that pairs of fermions can be created $(c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow})$ or annihilated $(c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow})$ out of and into a reservoir. This suggests that we should look for a ground state characterised by an uncertainty in the particle number. A widely used many-particle state with such property is the coherent state.

4 BCS ground state as a coherent state of Cooper pairs

The great intuition of Schrieffer in 1957 was to describe the ground state of the mean field hamiltonian H^{BCS} as a coherent state of electron pairs, known now as the BCS wavefunction.

$$|\Psi_0^{\text{BCS}}\rangle = \mathscr{C} \exp\left(\sum_{\mathbf{k}} \alpha_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}\right)|0\rangle = \tag{64}$$

$$= \mathscr{C} \prod_{\mathbf{k}} \exp\left(\alpha_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}\right)|0\rangle \tag{65}$$

$$= \mathscr{C} \prod_{\mathbf{k}} \left[1 + \alpha_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} + \frac{1}{2!}\left(\alpha_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}\right)^2 + \frac{1}{3!}\left(\alpha_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}\right)^3 + ...\right]|0\rangle \tag{66}$$

where $|0\rangle$ is the vacuum, $\alpha_{\mathbf{k}}$ a complex parameter and $\mathscr{C}$ is a normalisation constant to be determined. We notice that terms of the kind $c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}$ and higher order terms will always be zero. Therefore we can keep only the first terms of the expansion

$$|\Psi_0^{\text{BCS}}\rangle = \mathscr{C} \prod_{\mathbf{k}} \left(1 + \alpha_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}\right)|0\rangle \tag{67}$$

In order to find the normalisation constant, we calculate

$$\langle 0|(1 + \alpha^*_{\mathbf{k}} c_{\mathbf{k}\uparrow} c_{-\mathbf{k}\downarrow})(1 + \alpha_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow})|0\rangle = 1 + |\alpha_{\mathbf{k}}|^2 \tag{68}$$

Therefore we see that we obtain a normalised state by writing the coherent state as

$$|\Psi_0^{\text{BCS}}\rangle = \prod_{\mathbf{k}} \left(\frac{1}{1 + |\alpha_{\mathbf{k}}|^2} + \frac{\alpha_{\mathbf{k}}}{1 + |\alpha_{\mathbf{k}}|^2} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}\right)|0\rangle \tag{69}$$

typically written as

$$|\Psi_0^{\text{BCS}}\rangle = \prod_{\mathbf{k}} \left(u_{\mathbf{k}} + v_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{\text{-}\mathbf{k}\downarrow}\right)|0\rangle \tag{70}$$

$|v_{\mathbf{k}}|^2$ represents the probability that the pair state ($\mathbf{k}\uparrow, \text{-}\mathbf{k}\downarrow$) is occupied, while $|u_{\mathbf{k}}|^2$ represents the probability that the pair state ($\mathbf{k}\uparrow, \text{-}\mathbf{k}\downarrow$) is unoccupied. This pair state is not necessarily a bound Cooper pair.

Historically, Bardeen Cooper and Schrieffer proposed the above wave function as an educated guess (ansatz in the language of the practitioners) for the ground state of H^{BCS}, and determined $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ using a variational approach (J. Bardeen, L. N. Cooper and J. R. Schrieffer Phys. Rev. 108, 1175 (1957)). This is not the only example of a Nobel prize-winning educated guess: creativity is indeed important in science. We will now follow a less creative but more pedagogical route, starting from the diagonalisation of the hamiltonian with a canonical transformation. This approach will allows us to obtain the entire spectrum of excited states, giving us direct access to finite-temperature properties, that are not available if we limit ourselves to the study of the ground state.

5 Canonical transformation

The BCS hamiltonian

$$H^{\mathrm{BCS}} = \sum_{\mathbf{k}} \left[\xi(\mathbf{k})(c^\dagger_{\mathbf{k}\uparrow} c_{\mathbf{k}\uparrow} + c^\dagger_{\mathbf{k}\downarrow} c_{\mathbf{k}\downarrow}) - \Delta^*_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} - \Delta_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} \right]$$

is written in a form that involves only pairs of creation or annihilation operators. It can be therefore diagonalised, allowing us to uncover both the ground state obtained heuristically by Schrieffer and the spectrum of elementary excitations, especially relevant at finite temperatures. In matrix form we have

$$H^{\mathrm{BCS}} = \sum_{\mathbf{k}} \begin{pmatrix} c^\dagger_{\mathbf{k}\uparrow} & c_{-\mathbf{k}\downarrow} \end{pmatrix} \begin{pmatrix} \xi(\mathbf{k}) & \Delta_{\mathbf{k}} \\ \Delta^*_{\mathbf{k}} & -\xi(\mathbf{k}) \end{pmatrix} \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c^\dagger_{-\mathbf{k}\downarrow} \end{pmatrix} \tag{71}$$

We are looking to express it as

$$H^{\mathrm{BCS}} = \sum_{\mathbf{k}\sigma} E(\mathbf{k}) b^\dagger_{\mathbf{k}\sigma} b_{\mathbf{k}\sigma} \tag{72}$$

through a canonical transformation of the fermion operators $c_{\mathbf{k}}$ and $c^\dagger_{\mathbf{k}}$ into a new set of operators $b_{\mathbf{k}}$ and $b^\dagger_{\mathbf{k}}$. If we can express the hamiltonian in a diagonal form, the new operators will represent directly the elementary excitations of the system. A canonical transformation is a linear transformation that preserves the fermion anticommutation relations. This approach was originally proposed in 1958 by Bogoliubov (On a new method in the theory of superconductivity, Nuovo Cimento 7, 794 (1958)) and independently by Valatin (Comments on the theory of superconductivity, Nuovo Cimento 7, 843 (1958)). The choice of linear combination is guided by a physical argument: if our elementary excitation take fermions in and out of a many-body state constituted by bound pairs, exciting one particle leaves a partner behind, also in an excited state. Therefore, the elementary excitation of one particle also requires the annihilation of its partner. This leads us to the Bogoliubov-Valatin transformation

$$b_{\mathbf{k}\uparrow} = u^*_{\mathbf{k}} c_{\mathbf{k}\uparrow} - v_{\mathbf{k}} c^\dagger_{-\mathbf{k}\downarrow} \tag{73}$$

$$b^\dagger_{-\mathbf{k}\downarrow} = v^*_{\mathbf{k}} c_{\mathbf{k}\uparrow} + u_{\mathbf{k}} c^\dagger_{-\mathbf{k}\downarrow} \tag{74}$$

or in matrix form

$$\begin{pmatrix} b_{\mathbf{k}\uparrow} \\ b^\dagger_{-\mathbf{k}\downarrow} \end{pmatrix} = \begin{pmatrix} u^*_{\mathbf{k}} & -v_{\mathbf{k}} \\ v^*_{\mathbf{k}} & u_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c^\dagger_{-\mathbf{k}\downarrow} \end{pmatrix} \tag{75}$$

We require this transformation to be unitary, therefore

$$\begin{pmatrix} u^*_{\mathbf{k}} & -v_{\mathbf{k}} \\ v^*_{\mathbf{k}} & u_{\mathbf{k}} \end{pmatrix} = |u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1 \tag{76}$$

The unitary transformation can be inverted straightforwardly

$$\begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c^\dagger_{-\mathbf{k}\downarrow} \end{pmatrix} = \begin{pmatrix} u_{\mathbf{k}} & v_{\mathbf{k}} \\ -v^*_{\mathbf{k}} & u^*_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} b_{\mathbf{k}\uparrow} \\ b^\dagger_{\text{-}\mathbf{k}\downarrow} \end{pmatrix} = U \begin{pmatrix} b_{\mathbf{k}\uparrow} \\ b^\dagger_{\text{-}\mathbf{k}\downarrow} \end{pmatrix} \tag{77}$$

The form we have taken for the unitary transformation also guarantees its canonical property (anticommutator preserving). In order to verify this fact, we can directly compute $\{b^\dagger_{\mathbf{k}\uparrow}, b_{\mathbf{k}\uparrow}\} = 1$:

$$\{b^\dagger_{\mathbf{k}\uparrow}, b_{\mathbf{k}\uparrow}\} = \{u_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} - v^*_{\mathbf{k}} c_{-\mathbf{k}\downarrow}, u^*_{\mathbf{k}} c_{\mathbf{k}\uparrow} - v_{\mathbf{k}} c^\dagger_{-\mathbf{k}\downarrow}\} = \tag{78}$$

$$= u_{\mathbf{k}} u^*_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c_{\mathbf{k}\uparrow} - u_{\mathbf{k}} v_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} - v^*_{\mathbf{k}} u^*_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} + v^*_{\mathbf{k}} v_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c^\dagger_{-\mathbf{k}\downarrow} +$$

$$u^*_{\mathbf{k}} u_{\mathbf{k}} c_{\mathbf{k}\uparrow} c^\dagger_{\mathbf{k}\uparrow} - v_{\mathbf{k}} u_{\mathbf{k}} c^\dagger_{-\mathbf{k}\downarrow} c^\dagger_{\mathbf{k}\uparrow} - u^*_{\mathbf{k}} v^*_{\mathbf{k}} c_{\mathbf{k}\uparrow} c_{-\mathbf{k}\downarrow} + v_{\mathbf{k}} v^*_{\mathbf{k}} c^\dagger_{-\mathbf{k}\downarrow} c_{-\mathbf{k}\downarrow}$$

$$= |u_{\mathbf{k}}|^2 \{c_{\mathbf{k}\uparrow}, c^\dagger_{\mathbf{k}\uparrow}\} - u_{\mathbf{k}} v_{\mathbf{k}} \{c^\dagger_{\mathbf{k}\uparrow}, c^\dagger_{-\mathbf{k}\downarrow}\} - v^*_{\mathbf{k}} u^*_{\mathbf{k}} \{c_{-\mathbf{k}\downarrow}, c_{\mathbf{k}\uparrow}\} + |v_{\mathbf{k}}|^2 \{c_{-\mathbf{k}\downarrow}, c^\dagger_{-\mathbf{k}\downarrow}\} = 1$$

We can make use of the prescribed anticommutators:

$$\{c_{\mathbf{k}\uparrow}, c^\dagger_{\mathbf{k}\uparrow}\} = 1 \tag{79}$$

$$\{c^\dagger_{\mathbf{k}\uparrow}, c^\dagger_{-\mathbf{k}\downarrow}\} = 0 \tag{80}$$

$$\{c_{-\mathbf{k}\downarrow}, c_{\mathbf{k}\uparrow}\} = 0 \tag{81}$$

$$\{c_{-\mathbf{k}\downarrow}, c^\dagger_{-\mathbf{k}\downarrow}\} = 1 \tag{82}$$

to verify that indeed our new operators b describe fermions. Now we are ready to diagonalise the mean-field BCS hamiltonian

$$H^{\text{BCS}} = \sum_{\mathbf{k}} \begin{pmatrix} c^\dagger_{\mathbf{k}\uparrow} & c_{-\mathbf{k}\downarrow} \end{pmatrix} \begin{pmatrix} \xi(\mathbf{k}) & \Delta_{\mathbf{k}} \\ \Delta^*_{\mathbf{k}} & -\xi(\mathbf{k}) \end{pmatrix} \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c^\dagger_{-\mathbf{k}\downarrow} \end{pmatrix} \tag{83}$$

$$= \sum_{\mathbf{k}} \begin{pmatrix} c^\dagger_{\mathbf{k}\uparrow} & c_{-\mathbf{k}\downarrow} \end{pmatrix} UU^\dagger \begin{pmatrix} \xi(\mathbf{k}) & \Delta_{\mathbf{k}} \\ \Delta^*_{\mathbf{k}} & -\xi(\mathbf{k}) \end{pmatrix} UU^\dagger \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c^\dagger_{-\mathbf{k}\downarrow} \end{pmatrix}$$

$$= \sum_{\mathbf{k}} \begin{pmatrix} b^\dagger_{\mathbf{k}\uparrow} & b_{-\mathbf{k}\downarrow} \end{pmatrix} \begin{pmatrix} E(\mathbf{k})_\uparrow & 0 \\ 0 & -E(\mathbf{k})_\downarrow \end{pmatrix} \begin{pmatrix} b_{\mathbf{k}\uparrow} \\ b^\dagger_{-\mathbf{k}\downarrow} \end{pmatrix}$$

The coefficients of the Bogoliubov transformation can be calculated by solving the eigenvalues equation

$$\begin{pmatrix} \xi(\mathbf{k}) & \Delta_{\mathbf{k}} \\ \Delta^*_{\mathbf{k}} & -\xi(\mathbf{k}) \end{pmatrix} \begin{pmatrix} u_{\mathbf{k}} \\ v_{\mathbf{k}} \end{pmatrix} = \lambda^{\uparrow,\downarrow}_{\mathbf{k}} \begin{pmatrix} u_{\mathbf{k}} \\ v_{\mathbf{k}} \end{pmatrix} \tag{84}$$

$$\begin{pmatrix} \xi(\mathbf{k}) - \lambda^{\uparrow,\downarrow}_{\mathbf{k}} & \Delta_{\mathbf{k}} \\ \Delta^*_{\mathbf{k}} & -\xi(\mathbf{k}) - \lambda^{\uparrow,\downarrow}_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} u_{\mathbf{k}} \\ v_{\mathbf{k}} \end{pmatrix} = 0 \tag{85}$$

$$\det \begin{pmatrix} \xi(\mathbf{k}) - \lambda_{\mathbf{k}}^{\uparrow,\downarrow} & \Delta_{\mathbf{k}} \\ \Delta_{\mathbf{k}}^{*} & -\xi(\mathbf{k}) - \lambda_{\mathbf{k}}^{\uparrow,\downarrow} \end{pmatrix} = 0 \tag{86}$$

which has two solutions

$$\lambda_{\mathbf{k}}^{\uparrow,\downarrow} = \pm\sqrt{\xi(\mathbf{k})^2 + \Delta_{\mathbf{k}}^2} \tag{87}$$

Therefore we have found the eigenvalues (we take the physical positive solutions)

$$E_{\mathbf{k}}^{\uparrow,\downarrow} = \sqrt{\xi(\mathbf{k})^2 + \Delta_{\mathbf{k}}^2} \tag{88}$$

We can now express the BCS hamiltonian in the form

$$H^{\mathrm{BCS}} = \sum_{\mathbf{k}} \sqrt{\xi(\mathbf{k})^2 + \Delta_{\mathbf{k}}^2} \left(b_{\mathbf{k}\uparrow}^{\dagger} b_{\mathbf{k}\uparrow} + b_{\text{-}\mathbf{k}\downarrow}^{\dagger} b_{\text{-}\mathbf{k}\downarrow} \right) \tag{89}$$

In systems with the symmetries $\xi(\mathbf{k}) = \xi(\text{-}\mathbf{k})$ and $\Delta_{\mathbf{k}} = \Delta_{\text{-}\mathbf{k}}$ we arrive to the most simple diagonal formulation of the BCS hamiltonian

$$H^{\mathrm{BCS}} = \sum_{\mathbf{k}\sigma} \sqrt{\xi(\mathbf{k})^2 + \Delta_{\mathbf{k}}^2} \left(b_{\mathbf{k}\sigma}^{\dagger} b_{\mathbf{k}\sigma} \right) = \sum_{\mathbf{k}\sigma} E(\mathbf{k}) b_{\mathbf{k}\sigma}^{\dagger} b_{\mathbf{k}\sigma} \tag{90}$$

We now want to calculate the coefficients of the transformation, contained in the eigenvectors. Substituting the solutions into the eigenvalues equation we now find

$$\xi(\mathbf{k}) u_{\mathbf{k}} + \Delta_{\mathbf{k}} v_{\mathbf{k}} = u_{\mathbf{k}} \sqrt{\xi(\mathbf{k})^2 + \Delta_{\mathbf{k}}^2} \tag{91}$$

$$\Delta_{\mathbf{k}} u_{\mathbf{k}} - \xi(\mathbf{k}) v_{\mathbf{k}} = v_{\mathbf{k}} \sqrt{\xi(\mathbf{k})^2 + \Delta_{\mathbf{k}}^2} \tag{92}$$

Some algebra leads to

$$u_{\mathbf{k}} \left(\xi(\mathbf{k}) u_{\mathbf{k}} + \Delta_{\mathbf{k}} v_{\mathbf{k}} \right) = u_{\mathbf{k}} \left(u_{\mathbf{k}} \sqrt{\xi(\mathbf{k})^2 + \Delta_{\mathbf{k}}^2} \right) \tag{93}$$

$$v_{\mathbf{k}} \left(\Delta_{\mathbf{k}} u_{\mathbf{k}} - \xi(\mathbf{k}) v_{\mathbf{k}} \right) = v_{\mathbf{k}} \left(v_{\mathbf{k}} \sqrt{\xi(\mathbf{k})^2 + \Delta_{\mathbf{k}}^2} \right) \tag{94}$$

Substracting the two equations above we come to

$$\xi(\mathbf{k})(u_{\mathbf{k}}^2 + v_{\mathbf{k}}^2) = \sqrt{\xi(\mathbf{k})^2 + \Delta_{\mathbf{k}}^2}(u_{\mathbf{k}}^2 - v_{\mathbf{k}}^2) \tag{95}$$

Remembering that

$$u_{\mathbf{k}}^2 + v_{\mathbf{k}}^2 = 1 \tag{96}$$

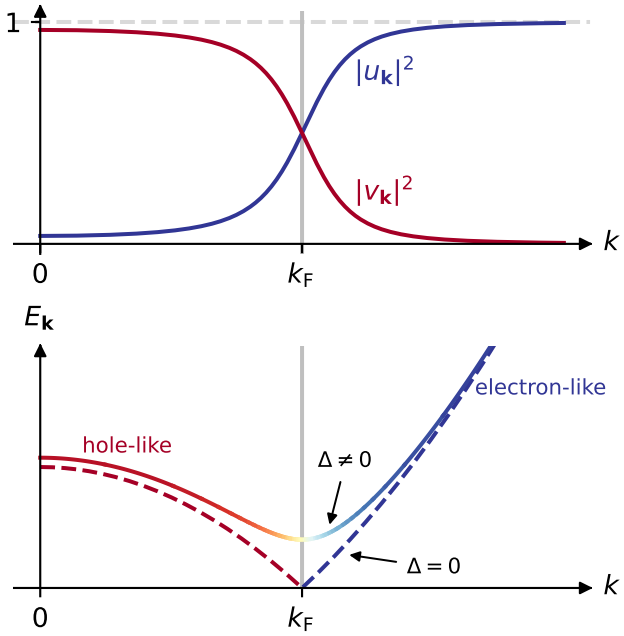

Figure 5: Excitation spectrum of the BCS mean field model. Excitations are electron-like above k_F and hole-like below k_F. When $\Delta \neq 0$ the excitations in the vicinity of the gapped region of the spectrum are a superposition of electron and hole.

we find

$$|u_{\mathbf{k}}|^2 = \frac{1}{2}\left(1 + \frac{\xi(\mathbf{k})}{\sqrt{\xi(\mathbf{k})^2 + \Delta_{\mathbf{k}}^2}}\right) \tag{97}$$

$$|v_{\mathbf{k}}|^2 = \frac{1}{2}\left(1 - \frac{\xi(\mathbf{k})}{\sqrt{\xi(\mathbf{k})^2 + \Delta_{\mathbf{k}}^2}}\right) \tag{98}$$

We saw before that $|v_{\mathbf{k}}|^2$ represents the probability that the pair state $(\mathbf{k}\uparrow, \text{-}\mathbf{k}\downarrow)$ is occupied, while $|u_{\mathbf{k}}|^2$ represents the probability that the pair state $(\mathbf{k}\uparrow, \text{-}\mathbf{k}\downarrow)$ is unoccupied.

Here we see that $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ also represent the weight of electron-like and hole-like character of the excitations of the BCS state (the Bogoliubov quasiparticles). $|u_{\mathbf{k}}|^2$ gives the probability that an excitation would be an electron if its charge is measured, and $|u_{\mathbf{k}}|^2$ gives the probability that an excitation would be a hole if its charge is measured.

6 Bogoliubov quasiparticles

Let's start by analysing the elementary excitations of the BCS state, known as Bogoliubov quasiparticles. These are not to be confused with Landau quasiparticle, the elementary excitations of the Fermi liquid. It is instructive to reflect on the physical meaning and some limiting cases of the excitation spectrum of the BCS mean-field hamiltonian. We will consider the example of elementary excitations of the kind $b^\dagger_{\mathbf{k}\uparrow}$. If we excite the system far above the Fermi energy (remember that in our definitions $\xi(\mathbf{k}) = 0$ at k_F) and far above the gap ($\mathbf{k}$ states for which $\xi(\mathbf{k}) \gg \Delta_{\mathbf{k}}$) we find

$$E(\mathbf{k}) \approx \xi(\mathbf{k}) \ , \ u_{\mathbf{k}} \approx 1 \ , \ v_{\mathbf{k}} \approx 0 \tag{99}$$

$$b^\dagger_{\mathbf{k}\uparrow} = u_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} - v^*_{\mathbf{k}} c_{-\mathbf{k}\downarrow} \approx c^\dagger_{\mathbf{k}\uparrow} \tag{100}$$

In this limiting case, the Bogoliubov quasiparticle looks a lot like an electron.

Similarly, if we excite the system far below the Fermi energy and far below the gap ($\mathbf{k}$ states for which $\xi(\mathbf{k}) \ll -|\Delta_{\mathbf{k}}|$) we find $u_{\mathbf{k}} \approx 0$, $v_{\mathbf{k}} \approx 1$. The Bogoliubov quasiparticle looks a lot like a hole.

The most spectacular effects are observed in the vicinity of the Fermi surface, close to the energy gap. Excitations precisely at $\mathbf{k}_F$ ($\xi(\mathbf{k}) = 0$) are characterised by

$$E(\mathbf{k}_F) = \Delta \ , \ u_{\mathbf{k}} = \frac{1}{\sqrt{2}} \ , \ v_{\mathbf{k}} = \frac{1}{\sqrt{2}} \tag{101}$$

$$b^\dagger_{\mathbf{k}\uparrow} = \frac{1}{\sqrt{2}} c^\dagger_{\mathbf{k}\uparrow} - \frac{1}{\sqrt{2}} c_{-\mathbf{k}\downarrow} \tag{102}$$

Bogoliubov quasiparticles in the vicinity of the gapped region of the spectrum are a superposition of electron and hole with the same amplitude.

7 BCS ground state as a vacuum of Bogoliubov quasiparticles

In the case of the Fermi gas, with hamiltonian

$$H^{\mathrm{FG}} = \sum_{\mathbf{k}\sigma} \epsilon(\mathbf{k}) c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} \tag{103}$$

we build the ground state $|\Psi^{\mathrm{FG}}_0\rangle$ from the vacuum $|0\rangle$ up to the Fermi wave vector $\mathbf{k}_F$ as

$$|\Psi^{\mathrm{FG}}_0\rangle = \prod_{\mathbf{k}\sigma}^{\mathbf{k}_F} c^\dagger_{\mathbf{k}\sigma}|0\rangle = \prod_{\mathbf{k}}^{\mathbf{k}_F} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{\text{-}\mathbf{k}\downarrow}|0\rangle \tag{104}$$

In analogy with the Fermi gas, in the case of the BCS hamiltonian

$$H^{\mathrm{BCS}} = \sum_{\mathbf{k}\sigma} E(\mathbf{k}) b^{\dagger}_{\mathbf{k}\sigma} b_{\mathbf{k}\sigma} \tag{105}$$

we want to build the ground state $|\Psi_0^{\mathrm{BCS}}\rangle$ from the vacuum $|0\rangle$ as

$$|\Psi_0^{\mathrm{BCS}}\rangle = \prod_{\mathbf{k}} b_{\mathbf{k}\uparrow} b_{\text{-}\mathbf{k}\downarrow} |0\rangle \tag{106}$$

This is an appropriate construction of the ground state because it builds a wave function for which the occupation number of Bogoliubov quasiparticles is guaranteed to be zero

$$\langle \Psi_0^{\mathrm{BCS}} | b^{\dagger}_{\mathbf{k}\sigma} b_{\mathbf{k}\sigma} | \Psi_0^{\mathrm{BCS}} \rangle = 0 \tag{107}$$

or equivalently that that $|\Psi_0^{\mathrm{BCS}}\rangle$ is a vacuum state of Bogoliubov quasiparticles

$$b_{\mathbf{k}\sigma} |\Psi_0^{\mathrm{BCS}}\rangle = 0 \tag{108}$$

In order to demonstrate this, we notice that

$$b_{\mathbf{k}'\sigma} |\Psi_0^{\mathrm{BCS}}\rangle = b_{\mathbf{k}'\sigma} \prod_{\mathbf{k}} b_{\mathbf{k}\uparrow} b_{\text{-}\mathbf{k}\downarrow} |0\rangle = 0 \tag{109}$$

for any $\mathbf{k}\sigma$, $b^2_{\mathbf{k}\sigma} = 0$.

Using the Bogoliubov-Valatin transformation

$$b_{\mathbf{k}\uparrow} = u^*_{\mathbf{k}} c_{\mathbf{k}\uparrow} - v_{\mathbf{k}} c^{\dagger}_{-\mathbf{k}\downarrow} \tag{110}$$

$$b_{-\mathbf{k}\downarrow} = v_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\uparrow} + u^*_{\mathbf{k}} c_{-\mathbf{k}\downarrow} \tag{111}$$

we can rewrite the ground state wave functions in terms of the original BCS formulation based on $c^{\dagger}_{\mathbf{k}\uparrow} c_{\text{-}\mathbf{k}\downarrow}$ operators

$$|\Psi_0^{\mathrm{BCS}}\rangle = \prod_{\mathbf{k}} b_{\mathbf{k}\uparrow} b_{\text{-}\mathbf{k}\downarrow} |0\rangle = \prod_{\mathbf{k}} \left(u^*_{\mathbf{k}} c_{\mathbf{k}\uparrow} - v_{\mathbf{k}} c^{\dagger}_{-\mathbf{k}\downarrow} \right) \left(v_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\uparrow} + u^*_{\mathbf{k}} c_{-\mathbf{k}\downarrow} \right) |0\rangle \tag{112}$$

$$= \prod_{\mathbf{k}} \left(u^*_{\mathbf{k}} v_{\mathbf{k}} c_{\mathbf{k}\uparrow} c^{\dagger}_{\mathbf{k}\uparrow} + |u_{\mathbf{k}}|^2 c_{\mathbf{k}\uparrow} c_{-\mathbf{k}\downarrow} - |v_{\mathbf{k}}|^2 c^{\dagger}_{\text{-}\mathbf{k}\downarrow} c^{\dagger}_{\mathbf{k}\uparrow} - v^*_{\mathbf{k}} u_{\mathbf{k}} c^{\dagger}_{\text{-}\mathbf{k}\downarrow} c_{\text{-}\mathbf{k}\downarrow} \right) |0\rangle$$

$$= \prod_{\mathbf{k}} \left(u^*_{\mathbf{k}} v_{\mathbf{k}} c_{\mathbf{k}\uparrow} c^{\dagger}_{\mathbf{k}\uparrow} - |v_{\mathbf{k}}|^2 c^{\dagger}_{\text{-}\mathbf{k}\downarrow} c^{\dagger}_{\mathbf{k}\uparrow} \right) |0\rangle$$

$$= \prod_{\mathbf{k}} \left(u^*_{\mathbf{k}} v_{\mathbf{k}} + |v_{\mathbf{k}}|^2 c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{\text{-}\mathbf{k}\downarrow} \right) |0\rangle$$

$$= \prod_{\mathbf{k}} v_{\mathbf{k}} \left(u_{\mathbf{k}}^* + v_{\mathbf{k}} c_{\mathbf{k}\uparrow}^\dagger c_{\text{-}\mathbf{k}\downarrow}^\dagger \right) |0\rangle$$

In order to normalise our ground state wave function it is useful to see that

$$= \langle 0 | (u_{\mathbf{k}} + v_{\mathbf{k}}^* c_{\text{-}\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow})(u_{\mathbf{k}}^* + v_{\mathbf{k}} c_{\mathbf{k}\uparrow}^\dagger c_{\text{-}\mathbf{k}\downarrow}^\dagger) | 0 \rangle \tag{113}$$

$$= \langle 0 | \left(|u_{\mathbf{k}}|^2 + u_{\mathbf{k}} v_{\mathbf{k}} c_{\mathbf{k}\uparrow}^\dagger c_{\text{-}\mathbf{k}\downarrow}^\dagger + v_{\mathbf{k}}^* u_{\mathbf{k}}^* c_{\text{-}\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} + |v_{\mathbf{k}}|^2 c_{\text{-}\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} c_{\mathbf{k}\uparrow}^\dagger c_{\text{-}\mathbf{k}\downarrow}^\dagger \right) | 0 \rangle$$

$$|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1$$

Therefore the normalised BCS ground state wavefunction is

$$|\Psi_0^{\text{BCS}}\rangle = \prod_{\mathbf{k}} \left(u_{\mathbf{k}}^* + v_{\mathbf{k}} c_{\mathbf{k}\uparrow}^\dagger c_{\text{-}\mathbf{k}\downarrow}^\dagger \right) |0\rangle \tag{114}$$

For $u_{\mathbf{k}}$ real, this expression becomes

$$|\Psi_0^{\text{BCS}}\rangle = \prod_{\mathbf{k}} \left(u_{\mathbf{k}} + v_{\mathbf{k}} c_{\mathbf{k}\uparrow}^\dagger c_{\text{-}\mathbf{k}\downarrow}^\dagger \right) |0\rangle \tag{115}$$

with

$$u_{\mathbf{k}} = \sqrt{\frac{1}{2}\left(1 + \frac{\xi(\mathbf{k})}{\sqrt{\xi(\mathbf{k})^2 + \Delta_{\mathbf{k}}^2}}\right)} \tag{116}$$

$$v_{\mathbf{k}} = \sqrt{\frac{1}{2}\left(1 - \frac{\xi(\mathbf{k})}{\sqrt{\xi(\mathbf{k})^2 + \Delta_{\mathbf{k}}^2}}\right)} \tag{117}$$

8 Gap equation

We want to calculate the gap function that we defined previously as

$$\Delta_{\mathbf{k}} = -\frac{1}{2\mathscr{V}} \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \langle c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} \rangle \tag{118}$$

We are interested in computing this thermodynamic average using the appropriate spectrum of eigenstates that we have already obtained. Therefore, we use again the Bogoliubov transformation in order to express the average explicitly as a function of our elementary excitations

$$\Delta_{\mathbf{k}} = -\frac{1}{2\mathscr{V}} \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \langle (-v_{\mathbf{k}'} b_{\mathbf{k}'\uparrow}^\dagger + u_{\mathbf{k}'} b_{-\mathbf{k}'\downarrow})(u_{\mathbf{k}'} b_{\mathbf{k}'\uparrow} + v_{\mathbf{k}'} b_{-\mathbf{k}'\downarrow}^\dagger) \rangle = \tag{119}$$

$$-\frac{1}{2\mathscr{V}}\sum_{\mathbf{k}'}V_{\mathbf{k}\mathbf{k}'}\left[-v_{\mathbf{k}'}u_{\mathbf{k}'}\langle b^\dagger_{\mathbf{k}'\uparrow}b_{\mathbf{k}'\uparrow}\rangle - v^2_{\mathbf{k}'}\langle b^\dagger_{\mathbf{k}'\uparrow}b^\dagger_{-\mathbf{k}'\downarrow}\rangle + u^2_{\mathbf{k}'}\langle b_{-\mathbf{k}'\downarrow}b_{\mathbf{k}'\uparrow}\rangle + u_{\mathbf{k}'}v_{\mathbf{k}'}\langle b_{-\mathbf{k}'\downarrow}b^\dagger_{-\mathbf{k}'\downarrow}\rangle\right]$$

We notice that our spectrum is formed by Bogoliubov fermions (characterised by a Fermi Dirac statistic), whose number operator is of the form $b^\dagger_{\mathbf{k}'\uparrow}b_{\mathbf{k}'\uparrow}$ with dispersion $E(\mathbf{k}) = \sqrt{\xi(\mathbf{k})^2+\Delta^2_{\mathbf{k}}}$. Therefore we see that

$$\langle b^\dagger_{\mathbf{k}'\uparrow}b^\dagger_{-\mathbf{k}'\downarrow}\rangle = 0 \tag{120}$$

$$\langle b_{-\mathbf{k}'\downarrow}b_{\mathbf{k}'\uparrow}\rangle = 0 \tag{121}$$

but thermal averaging of the number operator of our fermions will yield a Fermi-Dirac distribution $f(E(\mathbf{k}))$

$$\langle b^\dagger_{\mathbf{k}'\uparrow}b_{\mathbf{k}'\uparrow}\rangle = f(E(\mathbf{k}')) = \frac{1}{1+e^{E(\mathbf{k}')/kT}} \tag{122}$$

$$\langle b_{-\mathbf{k}'\downarrow}b^\dagger_{-\mathbf{k}'\downarrow}\rangle = 1-f(E(\mathbf{k}')) = 1-\frac{1}{1+e^{E(\mathbf{k}')/kT}} \tag{123}$$

Using this expression we come to the gap equation:

$$\Delta_{\mathbf{k}} = -\frac{1}{2\mathscr{V}}\sum_{\mathbf{k}'}V_{\mathbf{k}\mathbf{k}'}u_{\mathbf{k}'}v_{\mathbf{k}'}\left(1-2f(E(\mathbf{k}')\right) \tag{124}$$

We note that

$$u_{\mathbf{k}}v_{\mathbf{k}} = \frac{1}{2}\frac{\Delta_{\mathbf{k}}}{\sqrt{\xi(\mathbf{k})^2+\Delta^2_{\mathbf{k}}}} \tag{125}$$

Allowing us to write

$$\Delta_{\mathbf{k}} = -\frac{1}{2\mathscr{V}}\sum_{\mathbf{k}'}V_{\mathbf{k}\mathbf{k}'}\frac{1}{2}\frac{\Delta_{\mathbf{k}'}}{\sqrt{\xi(\mathbf{k}')^2+\Delta^2_{\mathbf{k}'}}}\left(1-2f(E(\mathbf{k}')\right) = -\frac{1}{2\mathscr{V}}\sum_{\mathbf{k}'}V_{\mathbf{k}\mathbf{k}'}\frac{\Delta_{\mathbf{k}'}}{2E(\mathbf{k}')}\left(1-2f(E(\mathbf{k}')\right) \tag{126}$$

We remember that the Fermi distribution can be expressed as

$$f(E(\mathbf{k}')) = \frac{1}{2}\left(1-\tanh\frac{E(\mathbf{k}')}{2kT}\right) \tag{127}$$

which leads to the nice and compact form of the gap equation

$$\Delta_{\mathbf{k}} = -\frac{1}{2\mathscr{V}}\sum_{\mathbf{k}'}V_{\mathbf{k}\mathbf{k}'}\frac{\Delta_{\mathbf{k}'}}{2E(\mathbf{k}')}\tanh\left(\frac{E(\mathbf{k}')}{2kT}\right) \tag{128}$$

Note that this is an equation of the kind $\Delta_{\mathbf{k}}(T) = F(\Delta_{\mathbf{k}}(T))$ and it needs to be solved numerically in order to obtain $\Delta(T)$. A qualitative form for the solution of a k-independent gap is shown in figure.

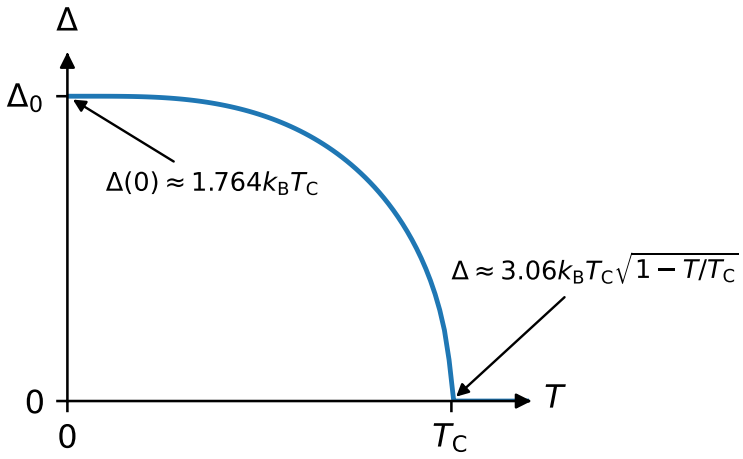

Figure 6: Temperature dependence of the gap

Importantly, this equation will have solutions only for $V < 0$, that is attractive interactions (self-consistency condition). This equation is often rewritten as the linearised gap equation, assuming the Cooper approximation for the form of the attractive interaction discussed above (also known as the BCS one-square-well model, see figure 3)

$$\Delta = -\frac{1}{2\mathscr{V}} V \Delta \sum_{\mathbf{k}'} \frac{1}{2E(\mathbf{k}')} \tanh\left(\frac{E(\mathbf{k}')}{2kT}\right) \tag{129}$$

$$1 = -\frac{1}{2\mathscr{V}} V \sum_{\mathbf{k}'} \frac{1}{2E(\mathbf{k}')} \tanh\left(\frac{E(\mathbf{k}')}{2kT}\right) \tag{130}$$

which in the continuous limit reads

$$1 = -V \int_{-\hbar\omega_D}^{\hbar\omega_D} dE \frac{g(E)}{2E} \tanh\left(\frac{E}{2kT}\right) \tag{131}$$

$$1 = -g(E_F)V \int_{-\hbar\omega_D}^{\hbar\omega_D} dE \frac{1}{2E} \tanh\left(\frac{E}{2kT}\right) = -g(E_F)V \ln\left(\frac{1.14\hbar\omega_D}{kT}\right) \tag{132}$$

$$kT_c = 1.14\hbar\omega_D e^{-1/g(E_F)V} = 1.14\hbar\omega_D e^{-1/\lambda} \tag{133}$$

We have come to a fundamental result: a relationship between material dependent parameters that identifies the superconducting critical temperature from a mean field self-consistency condition. The same approach can be utilised in describing other second order phase transitions, such as the mean field analysis of the Curie transition for ferromagnets.

At $T = 0$ we find

$$1 = -g(E_F)V \int_0^{\hbar\omega_D} dE \frac{1}{\sqrt{E^2 + \Delta^2}} = -g(E_F)V \sinh \frac{\hbar\omega_D}{\Delta} \tag{134}$$

$$\Delta \approx 2\hbar\omega_D e^{-1/g(E_F)V} = 1.764 kT_C \tag{135}$$

In a model that approximates more closely the effective electron-electron interaction (two-square-well model, see figure 3), by parametrizing the screened Coulomb interaction $\mu^\star$ we find

$$kT_c = 1.14\hbar\omega_D e^{-\frac{1}{\lambda - \mu^\star}} \tag{136}$$

with

$$\mu^\star = \frac{\mu}{1 + \mu \ln\left(\frac{E_F}{\hbar\omega_D}\right)} \tag{137}$$

and

$$\mu = g(E_F)V_{Coulomb} \tag{138}$$

In the BCS model we have found

$$T_c \propto \omega_D \propto M^{-1/2} \tag{139}$$

with M the isotopic mass. We define the isotope effect parameter α as

$$\alpha = -\frac{d \ln T_c}{d \ln M} \tag{140}$$

which can be experimentally measured. In the simple BCS model we would therefore have $\alpha = 1/2$. Using the two-square model, we find that this is an upper limit. Indeed we see

$$\ln T_c = \text{const} + \ln \hbar\omega_D - \frac{1}{\lambda - \mu^\star} \tag{141}$$

leading to

$$\alpha = \frac{1}{2}\left[1 - \left(\frac{\mu^\star}{\lambda - \mu^\star}\right)\right] \tag{142}$$

9 BCS density of states

The density of states of Bogoliubov quasiparticles is calculated using the expression:

$$g^{\text{BCS}}(E) = \frac{1}{\mathscr{V}} \sum_{\mathbf{k}} \delta(E - E(\mathbf{k})) = \frac{1}{\mathscr{V}} \sum_{\mathbf{k}} \delta\left(E - \sqrt{\xi^2(\mathbf{k}) + \Delta_{\mathbf{k}}^2}\right) \tag{143}$$

In the continuous limit and and for a k-independent gap this reads

$$g^{\mathrm{BCS}}(E) = \int_{-\infty}^{+\infty} g^{\mathrm{n}}(E)\delta\left(E - \sqrt{\xi^2 + \Delta^2}\right) d\xi \tag{144}$$

where $g^{\mathrm{n}}(E)$ is the density of states in the normal state. We use again the approximation of quasi-constant $g^{\mathrm{n}}(E)$ around the Fermi energy, to write

$$g^{\mathrm{BCS}}(E) \simeq g^{\mathrm{n}}(E_F) \int_{-\infty}^{+\infty} \delta\left(E - \sqrt{\xi^2 + \Delta^2}\right) d\xi \tag{145}$$

In order to compute this quantity we need to use the following property of the δ distribution

$$\delta\left[f(x)\right] = \sum_{x_i} \frac{\delta(x - x_i)}{|\frac{df(x_i)}{dx}|} \tag{146}$$

where the sum extends over all the solutions x_i of the equation $f(x) = 0$. In our current case we have

$$\delta\left[f(\xi)\right] = \delta\left[E - \sqrt{\xi^2 + \Delta^2}\right] \tag{147}$$

$$\frac{df}{d\xi} = -\frac{\xi}{\sqrt{\xi^2 + \Delta^2}} \tag{148}$$

The solutions of $f(\xi) = 0$ are $\xi_{1,2} = \pm\sqrt{E^2 - \Delta^2}$ We can then write

$$\delta\left[f(x)\right] = \frac{\delta\left(\xi - \sqrt{E^2 - \Delta^2}\right)}{\frac{\sqrt{E^2-\Delta^2}}{E}} + \frac{\delta\left(\xi + \sqrt{E^2 - \Delta^2}\right)}{\frac{\sqrt{E^2-\Delta^2}}{E}} \tag{149}$$

$$g^{\mathrm{BCS}}(E) \simeq g^{\mathrm{n}}(E_F)\frac{E}{\sqrt{E^2 - \Delta^2}} \int_{-\infty}^{+\infty} \delta\left(\xi - \sqrt{E^2 - \Delta^2}\right) + \delta\left(\xi + \sqrt{E^2 - \Delta^2}\right) d\xi \tag{150}$$

$$= g^{\mathrm{n}}(E_F)\frac{2E}{\sqrt{E^2 - \Delta^2}} \quad \text{if} \quad |E| > \Delta \tag{151}$$

$$= 0 \quad \text{if} \quad -\Delta < E < \Delta$$

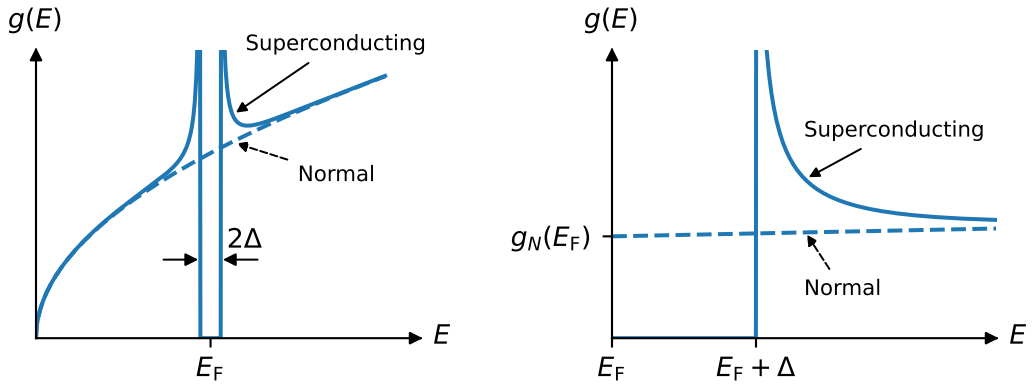

Figure 7: Gapped density of states at T=0